ISSN 2277-3[illegible]X

JOURNAL OF PURE MATHEMATICS

Volume 28, 2011

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF CALCUTTA
35, Ballygunge Circular Road, Kolkata - 700 019, India

JOURNAL OF PURE MATHEMATICS
UNIVERSITY OF CALCUTTA

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Kolkata 700 019, India
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ISSN 2277-355X

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Jour. Pure Math., Vol. 28, 2013, pp. 1-18

METHODS OF SOLVING SYSTEMS OF LINEAR FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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ABSTRACT : Three methods of finding solutions of systems of linear first order ordinary differential equations (SODEs) with variable coefficients have been developed. The methods have been illustrated by examples.

Key words : Systems of linear first-order ordinary differential equations with variable coefficients, systems of linear algebraic equations, exact differential equations.
AMS Classification : 34B.

1. INTRODUCTION

A method of solving linear first-order systems of ordinary differential equations with constant coefficients has been developed in [1]; the present article extends the idea developed there to SODEs with variable coefficients.

The SODEs under consideration comprises linear first order ordinary differential equations of the form

$$M_i(x) = \sum_{j=1}^{n} l_{ij}(t)x_j'(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t) = b_i(t), \ (i = 1, 2, ..., n), \tag{1.1}$$

where $t \in I$ (interval), $x = x(t) = (x_1(t), x_2(t), ..., x_n(t)^{\mathrm{T}}$,

$\ell_{ij}, a_{ij} : \mathrm{I} \to \mathbf{C}$ (set of complex numbers) $(i, j = 1, 2, ...n)$ are continuous, $' \equiv \frac{d}{dt}$ and A^T denotes the transpose of the matrix A.

Writing $L(t) = (\ell_{ij}(t)_{n\times n}, \quad A(t) = (a_{ij}(t))_{n\times n}$,

$X(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $X'(t) = (x'_1(t), x'_2(t), \ldots, x'_n(t))^T$, the homogeneous SODEs corresponding to the SODEs (1.1) can be exhibited as

$$L(t)X'(t) = A(t)X(t), \qquad (t \in I). \tag{1.2}$$

To find the solutions of the SODEs (1.1) it is enough to know only one solution of the SODEs (1.1), provided all solutions of the corresponding homogeneous SODEs (1.2) are known. Henceforth our attention is therefore confined to finding the solutions of the homogeneous SODEs (1.2).

Three approaches for solving the homogeneous SODEs (1.2) have been discussed in the sequel :

I. By deriving exact differential equations from the given SODEs (1.2).

II. By reducing the given SODEs (1.2) with the help of the adjoint matrices L^*, A^*.

III. By using integrating factors obtainable from another SODEs corresponding to the given SODEs (1.2), called its adjoint SODEs.

Having obtained a set of linearly independent solutions $\varphi_1, \varphi_2, \ldots, \varphi_n$ of the homogeneous SODEs (1.2), a solution ψ of the nonhomogeneous SODEs (1.1) can be derived by the method of variation of parameters. All solutions of the nonhomogeneous SODEs (1.1) are then expressible as $\psi + C_1\varphi_1 + C_2\varphi_2 + \ldots + C_n\varphi_n$, with suitable $C_1, C_2.., C_n$ $(\in \mathbf{C})$. Hence, only the approaches, I, II, III are elaborated in the sequel.

2. APPROACH I : BY DERIVING EXACT DIFFERENTIAL EQUATIONS FROM THE GIVEN SODES (1.2)

The case $n = 2$ is presented below in detail; the case $n > 2$ can be treated similarly. For abbreviation let us write in (1.2).

$$\ell_{ij}(t) = \ell_{ij}, \quad a_{ij}(t) = a_{ij}, \quad x_i(t) = x_i \quad (i, j = 1, 2, \ldots, n).$$

Writing

$$\underline{r} = (x_1, x_2)^T, \ \underline{r}' = (x'_1, x'_2)^T, \ \underline{L}_i = (\ell_{i1}, \ell_{i2})^T, \ \underline{A}_i = (a_{i1}, a_{i2})^T, \quad (i = 1, 2) \tag{2.1}$$

the equations of the SODEs (1.2) can be exhibited as

$$\underline{r}'^{T}.\underline{L}_i = \underline{r}^{T}.\underline{A}_i, \quad (i = 1, 2) \quad (t \in I). \tag{2.2}$$

Two cases arise :

Ia. : Each equation of (2.2) can be converted to an exact differential equation.

Ib. : Linear combinations of the equations of (2.2) can be so determined that the resulting equation becomes convertible to an exact differential equation.

Case Ia :

The first equation of (2.2) can be rewritten as

$$(\underline{r}^{T}.\underline{L}_1)' = \underline{r}^{T}.(\underline{A}_1 + L'_1), \quad (t \in I). \tag{2.3}$$

If there exists $f_1 : I \to C$ such that

$$\underline{A}_1 + \underline{L}'_1 = f_1\underline{L}_1, \quad (t \in I), \tag{2.4}$$

then (2.3) becomes

$$(\underline{r}^{T}.\underline{L}_1)' = f_1(\underline{r}^{T}.\underline{L}_1), \quad (t \in I), \tag{2.5}$$

which can be converted to an exact differential equation.

On integration one gets from (2.5)

$$r^{T}.L_1 = C_1 \exp\left[\int_{\alpha_1}^{t} f_1(s)\,ds\right], \quad (t \in I), \tag{2.6}$$

where α_1 is an arbitrarily chosen point of I and C_1 ($\in \mathbf{C}$) is the parameter of integration.

Proceeding similarly with the second equation of (2.2), if there exists $f_2 : I \to \mathbf{C}$ such that

$$\underline{A}_2 + \underline{L}'_2 = f_2\underline{L}_2, \quad (t \in I), \tag{2.7}$$

the second equation of (2.2) becomes

$$(\underline{r}^{T}.\underline{L}_2)' = f_2(\underline{r}^{T}.\underline{L}_2), \quad (t \in I). \tag{2.8}$$

Integrating (2.8) one obtains

$$\underline{\mathrm{r}}^T.\underline{\mathrm{L}}_2 = C_2 \exp\left[\int_{\alpha_2}^{t} f_2(s)ds\right], \quad (t \in I), \tag{2.9}$$

where α_2 is an arbitrarily chosen point of I and C_2 ($\in$ **C**) is the parameter of integration.

The required solutions of the SODEs (1.2) are then obtained by solving the two linear algebraic equations (2.6) and (2.9).

Note : The success of the above procedure clearly depends upon the existence of the functions $f_i : I \to \mathbf{C}$ (i = 1, 2) satisfying respectively (2.4) and (2.7).

Assuming the existence of the functions f_1, f_2 one gets

$$f_1 = (a_{11} + \ell'_{11})/\ell_{11} = (a_{12} + \ell'_{12})/\ell_{12}, \quad (t \in I), \tag{2.10}$$

$$f_2 = (a_{21} + \ell'_{21})/\ell_{21} = (a_{22} + \ell'_{22})/\ell_{22}, \quad (t \in I), \tag{2.11}$$

provided none of the ℓ_{ij}'s (i, j = 1, 2) is zero.

Conversely, if (2.10) ((2.11)) holds, the existence of $f_1(f_2)$ is guaranteed.

The above observation establishes the following result :

If $\ell_{i1}\ \ell_{i2} \neq 0$, the ith equation of the SODEs (1. 2) with $n = 2$ is convertible to an exact differential equaton if and only if

$$(a_{i1} + \ell'_{i1})/\ell_{i1} = (a_{i2} + \ell'_{i2})/\ell_{i2}, \quad (t \in I), \quad (i = 1, 2).$$

This result can be easily extended to the case $n > 2$.

Example 1 :

$$tx'_1 + t^2x'_2 = -2x_1 - 3tx_2, \tag{ia}$$

$$t^2x'_1 + tx'_2 = -tx_1, \tag{ib}$$

where x_i :]0, 1[$\to \mathbf{R}$ (set of real numbers), i = 1, 2.

It can be easily verified that the conditions (2.10), and (2.11) are satisfied here.

Actually (ia) can be rewritten as

$$\left(tx_1 + t^2x_2\right)' = -\frac{1}{t}\left(tx_1 + t^2x_2\right), \quad t \in]0,\ 1[. \tag{ii}$$

Hence, on integration, one obtains

$$tx_1 + t^2x_2 = C_1/t, \quad t \in]0,\ 1[, \tag{iii}$$

where C_1 ($\in$ **R**) is the parameter of integration.

Similarly (ib) can be rewritten as

$$\left(t^2x_1 + tx_2\right)' = \frac{1}{t}\left(t^2x_1 + tx_2\right), \quad t \in]0,\ 1[. \tag{iv}$$

On integration (iv) yields

$$t^2x_1 + tx_2 = C_2t, \quad t \in]0,\ 1[, \tag{v}$$

where C_2 ($\in$ **R**) is the parameter of integration.

The required solutions of the SODEs (ia) – (ib) are obtained by solving the algebraic equations (iii) and (v).

Case Ib :

In case Ia, each of the equations of (1.2) has been assumed to be reducible to an exact ordinary differential equation [vide (2,5), (2.8)].

If such remodelling of the individual equation of (1.2) to convert it to an exact ODE is not possible, one may look for functions $\lambda_i : I \to \mathbf{C}$ ($i = 1, 2$), so that the ODE

$$\lambda_1(\underline{r}'^T.\underline{L}_1) + \lambda_2(\underline{r}'^T.\underline{L}_2) = \lambda_1(\underline{r}^T.\underline{A}_1) + \lambda_2(\underline{r}^T.\underline{A}_2), \quad (t \in I), \tag{2.12}$$

can be reduced to an exact ODE.

Now (2.12) is rewritten as

$$\underline{r}'^T.(\lambda_1\underline{L}_1 + \lambda_2\underline{L}_2) + \underline{r}^T.(\lambda_1\underline{L}_1 + \lambda_2\underline{L}_2)' = [\underline{r}^T.(\lambda_1\underline{L}_1 + \lambda_2\underline{L}_2)]'$$

$$= \underline{r}^T.[(\lambda_1\underline{L}_1 + \lambda_2\underline{L}_2)' + (\lambda_1\underline{A}_1 + \lambda_2\underline{L}_2)], \quad (t \in I). \tag{2.13}$$

Comparing (2.13) with (2.3) one derives that (2.13) is convertible to an exact ODE if and only if there exists $g : I \to \mathbf{C}$ such that

$$(\lambda_1 \underline{L}_1 + \lambda_2 \underline{L}_2)' + (\lambda_1 \underline{A}_1 + \lambda_2 \underline{A}_2) = g(\lambda_1 \underline{L}_1 + \lambda_2 \underline{L}_2), \quad (t \in I). \tag{2.14}$$

The solutions of the SODEs (2.14) will provide the requisite λ_1, λ_2 for making (2.12) an exact ODE.

It is further observed that (2.14) can be rewritten as

$$(\mu_1 \underline{L}_1 + \mu_2 \underline{L}_2)' + (\mu_1 \underline{A}_1 + \mu_2 \underline{A}_2) = 0, \tag{2.15}$$

where $\mu_i(t) = \lambda_i(t) \exp\left[-\int_\alpha^t g(s)\,ds\right], \quad (t \in I), \quad (i = 1, 2),$ (2.16)

$\alpha (\in I)$ being fixed arbitrarily.

The equation (2.12) being homogeneous in λ_1 and λ_2, it is enough to find $\lambda_1 : \lambda_2$; and so $\mu_1 : \mu_2$ will equivalently serve the purpose.

To derive $\mu_1 : \mu_2$ from (2.15) it is noted that, (2.15) becomes integrable if there exists $h : I \to \mathbf{C}$ such that

$$\mu_1 \underline{A}_1 + \mu_2 \underline{A}_2 = h[\mu_1 \underline{L}_1 + \mu_2 \underline{L}_2], \quad (t \in I). \tag{2.17}$$

Notably, for the two scalar equations of the vector equation (2.17), it is necessary to have the same function h, otherwise the scalar differential equation (2.13) cannot be solved. The two equations of (2.17) are

$$\mu_1(a_{11} - h\ell_{11}) + \mu_2(a_{21} - h\ell_{21}) = 0, \quad (t \in I), \tag{2.18a}$$

$$\mu_1(a_{12} - h\ell_{12}) + \mu_2(a_{22} - h\ell_{22}) = 0, \quad (t \in I). \tag{2.18b}$$

(2.18a) – (2.18b) yield $(\mu_1, \mu_2) \neq (0, 0)$ if

$$\begin{vmatrix} a_{11} - h\ell_{11} & a_{21} - h\ell_{21} \\ a_{12} - h\ell_{12} & a_{22} - h\ell_{22} \end{vmatrix} = 0, \quad (t \in I). \tag{2.19}$$

(2.19) is a quadratic in h, and so yields, in general, two functions h_1, h_2 say. Substituting $h_i(i = 1, 2)$ for h in (2.18a) – (2.18b), the resulting algebraic equations in μ_1, μ_2 are solved : let the solution of (2.18a) – (2.18b), with $h = h_i$, be denoted by μ_{i1}, μ_{i2} $(i = 1, 2)$.

It is to be noted carefully that the μ_{i1}, μ_{i2} thus obtained may or may not satisfy the SODEs (2.15) for μ_1, μ_2, because the existence of a function h satisfying (2.17) is not guaranteed by the integrability of the SODEs (2.15).

Supposing μ_{i1}, μ_{i2} satisfy (2.15), one obtains

$$[\mu_{i1}\underline{L}_1 + \mu_{i2}\underline{L}_2]' + h_i[\mu_{i1}\underline{L}_1 + \mu_{i2}\underline{L}_2] = 0, \quad (t \in I), \quad (i = 1, 2). \tag{2.20}$$

Hence, integrating (2.20) one gets

$$\mu_{i1}\underline{L}_1 + \mu_{i2}\underline{L}_2 = (C_{i1}, C_{i2})^T \exp\left[-\int_a^t h_i(s)\,ds\right], \quad (t \in I), \quad (i = 1, 2), \tag{2.21}$$

where C_{i1}, C_{i2} $(\in \mathbf{C})$ are the parameters of integration, $\alpha(\in I)$ being chosen arbitrarily. Using $\mu_{i1} : \mu_{i2}$ for $\mu_1 : \mu_2 = \lambda_1 : \lambda_2$ in (2.12) one obtains

$$\underline{r}'^{T}.(\mu_{i1}\underline{L}_1 + \mu_{i2}\underline{L}_2) = r^T.(\mu_{i1}\underline{A}_1 + \mu_{i2}\underline{A}_2) = h_i\underline{r}^{T}.(\mu_{i1}\underline{L}_1 + \mu_{i2}\underline{L}_2), \quad (i = 1\ 2),$$

which, on using (2.21), become

$$\underline{r}'^{T}.(C_{i1}, C_{i2})^T = \underline{r}^T.h_i(C_{i1}, C_{i2})^T$$

i.e. $C_{i1}x'_1 + C_{i2}x'_2 = h_i(C_{i1}x_1 + C_{i2}x_2), \quad (t \in I), \quad (i = 1, 2).$

Hence, $$C_{i1}x_1 + C_{i2}x_2 = D_i \exp\left[\int_a h_i(s)\,ds\right], \quad (t \in I), \quad (i = 1, 2). \tag{2.22}$$

where D_i $(\in \mathbf{C})$ $(i = 1, 2)$ are the parameters of integration.

The required solutions of the SODEs (2.2) are then obtained on solving the algebraic equations (2.22).

Example 2 : $x'_1 + tx'_2 = tx_1 + t^3x_2,$ (i*a*)

$$x'_1 + t^2x'_2 = tx_1 + t^4x_2, \qquad \text{(ib)}$$

where $x_i : \mathbb{R} \to \mathbb{R}, \quad i = 1, 2.$

In this case (2.19) becomes

$$\begin{vmatrix} t-h & t-h \\ t^3 - th & t^4 - t^2h \end{vmatrix} = 0.$$

Hence $h = t$ or t^2.

For $h = t$, (2.18*a*) becomes an identity and (2.18*b*) becomes $\mu_1 + \mu_2 t = 0$.

So, $\lambda_1 : \lambda_2 = \mu_1 : \mu_2 = t : -1.$ (iii)

Actually, subtracting (ib) from t-times (ia), one gets $x'_1 = tx_1$.

Hence $x_1 = C_1 \exp(t^2/2), \quad (t \in \mathbb{R}),$ (iv)

where $C_1 (\in \mathbb{R})$ is the parameter of integration.

For $h = t^2$, (2.18b) becomes an identity, and (2.18a) becomes $\mu_1 + \mu_2 = 0$.

So, $\lambda_1 : \lambda_2 = \mu_1 : \mu_2 = 1 : -1.$ (v)

Subtracting (ib) from (ia) one gets $x'_2 = t^2x_2 \quad (t \in \mathbb{R}).$

Hence, $x_2 = C_2 \exp(t^3/3), \quad (t \in \mathrm{R}),$ (vi)

where $C_2 \ (\in \mathbb{R})$ is the parameter of integration.

(iv) and (vi) provide the required solution.

3. APPROACH II : BY REDUCING THE GIVEN SODES WITH THE HELP OF THE ADJOINT MATRICES L^*, A^*

The homogeneous SODEs (1.2) corresponding to the given SODEs (1.1) can be represented in matrix form as

$$L(t)\ X'(t) = A(t)\ X(t), \quad (t \in I). \tag{3.1}$$

If there exists a matrix $P(t) = (p_{ij}(t))_{n\times n}$, $(t \in I)$ such that both $P(t)L(t)$ and $P(t)A(t)$ become diagonal matrices:

$$P(t)L(t) = \text{diag}\ (\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)), \quad (t \in I), \tag{3.2}$$

$$P(t)A(t) = \text{diag}(\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)), \quad (t \in I), \tag{3.3}$$

then, multiplying both sides of (3.1) by $P(t)$ from left, one obtains

$$\begin{aligned}&[\text{diag}(\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t))]X'(t),\\ &\quad = [\text{diag}\ (\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))]X(t), \quad (t \in I).\end{aligned} \tag{3.4}$$

Hence, equating the corresponding components from both sides of (3.4) one gets

$$\lambda_k(t)\, x'_k(t) = \alpha_k(t) x_k(t), \quad (t \in I), \quad k = 1, 2, \ldots, n. \tag{3.5}$$

Therefore, on integration, one derives

$$x_k(t) = C_k \exp\left(\int_a^t \alpha_k(u) / \lambda_k(u) du\right), \quad k = 1, 2, \ldots n, \tag{3.6}$$

where $C_k (\in \mathbf{C})$, $k = 1, 2, \ldots, n$, are the parameters of integration, and "a" ($\in I$) is arbitrarily chosen, provided $\lambda_k(t) \neq 0$, $k = 1, 2, \ldots, n$, $(t \in I)$.

Thus, the problem of finding the solutions of the SODEs (1.2) is reduced to the problem of finding a matrix $P(t)$ that satisfies (3.2) and (3.3). To determine such a matrix $P(t)$ one observes that (3.2) implies

$$\sum_{j=1}^{n} p_{ij}(t)\ell_{jk}(t) = \delta_{ik}\lambda_k(t), \quad (i,\ k = 1,\ 2,\ \ldots,\ n), \quad (t \in I), \tag{3.7}$$

where δ_{ik} denotes the Kronecker's delta.

If $L_{ij}(t)$ denotes the cofactor of $\ell_{ij}(t)$ in $L(t) = (\ell_{ij}(t))_{n\times n}$, it can be derived from (3.7) that

$$\frac{p_{i1}(t)}{L_{1i}(t)} = \frac{p_{i2}(t)}{L_{2i}(t)} = \ldots = \frac{p_{in}(t)}{L_{ni}(t)}, \quad i = 1,\ 2,\ \ldots,\ n, \quad (t \in I). \tag{3.8}$$

Dealing similarly with the equation (3.3) one obtains

$$\frac{p_{i1}(t)}{A_{1i}(t)} = \frac{p_{i2}(t)}{A_{2i}(t)} = \ldots = \frac{p_{in}(t)}{A_{ni}(t)}, \quad i = 1,\ 2,\ \ldots,\ n, \quad (t \in I). \tag{3.9}$$

From (3.8) and (3.9) it follows that, if a matrix $P(t)$ exists, that satisfies (3.2) and (3.3), then the following must hold good :

$$\frac{L_{1i}(t)}{A_{1i}(t)} = \frac{L_{2i}(t)}{A_{2i}(t)} = \ldots = \frac{L_{ni}(t)}{A_{ni}(t)}, \ (= g_i(t), \text{ say}), \quad i = 1,\ 2,\ \ldots,\ n, \quad (t \in I). \tag{3.10}$$

Conversely, if (3.10) holds good, then the matrix

$$Q(t) = (f_i(t)\ L_{ji}(t))_{n\times n}, \qquad (t \in I), \tag{3.11}$$

(where $f_i : I \to \mathbf{C}$, $i = 1, 2, \ldots, n$, are arbitrary functions) satisfies (3.2) and (3.3), when substituted for $P(t)$. The choice $f_i(t) = 1$, $i = 1, 2, \ldots, n$, $(t \in I)$ renders $Q(t)$ to be $L^*(t)$, the adjoint of the matrix $L(t)$.

Hence, **(3.10) represents the necessary and sufficient conditions for the existence of a matrix P(t) satisfying (3.2) and (3.3).**

Multiplying (3.1) by $Q(t)$ from left one gets, using (3.10),

$$(f_i(t)L_{ji}(t\ ))\ (\ell_{ij}(t))X'(t) = (f_i(t)g_i(t)A_{ji}(t))\ (a_{ij}(t))X(t), \qquad (t \in I),$$

which immediately reduces to

$$[\mathrm{diag}(f_1(t)\ |L|,\ f_2(t)\ |L|,\ \ldots,\ f_n(t)\ |L|)]\ X'\ (t)$$

$$= [\mathrm{diag}(f_1(t)\, g_1(t)\ |A|,\ f_2(t) g_2(t)\, |A|,\ \ldots,\ f_n(t)\, g_n(t)\ |A|)]\, X(t),\ (t \in I), \tag{3.12}$$

where $|L|$ and $|A|$ denote respectively the determinant of the matrix L and A.

Equating the corresponding components of (3.12) one gets

$$|L| x'_k(t) = g_k(t)\, |A|\, x_k(t), \quad k = 1, 2, \ldots, n, \quad (t \in I). \tag{3.13}$$

The required solutions of the homogeneous SODEs (1.2) are then obtained from (3.13) on integration.

Example 3 : $x_1' + tx_2' + t^2x_3' = \frac{1}{t}x_1 + t^3x_2 + t^4x_3,$ (ia)

$$tx_1' + t^2x_2' + x_3' = x_1 + t^4x_2 + t^2x_3, \tag{ib}$$

$$t^2x_1' + x_2' + tx_3' = tx_1 + t^2x_2 + t^3x_3, \tag{ic}$$

where $x_k : I = \mathbf{R} \setminus \{0\} \to \mathbf{R},\ k = 1, 2, 3.$

$$\text{Let L} = \begin{bmatrix} 1 & t & t^2 \\ t & t^2 & 1 \\ t^2 & 1 & t \end{bmatrix} \quad \text{and} \quad \text{A} = \begin{bmatrix} \frac{1}{t} & t^3 & t^4 \\ 1 & t^4 & t^2 \\ t & t^2 & t^3 \end{bmatrix}$$

$$\text{Then L}^* = \begin{bmatrix} t^3 - 1 & 0 & t - t^4 \\ 0 & t - t^4 & t^3 - 1 \\ t - t^4 & t^3 - 1 & 0 \end{bmatrix} \quad \text{and A}^* = \begin{bmatrix} t^4(t^3 - 1) & 0 & t^4\left(t - t^4\right) \\ 0 & t\left(t - t^4\right) & t\left(t^3 - 1\right) \\ t\left(t - t^4\right) & t\left(t^3 - 1\right) & 0 \end{bmatrix}.$$

Writing (ia) – (ib) – (ic) as $LX' = AX$, where $X = (x_1, x_2, x_3)^T$, one obtains $\mathrm{L^*LX' = L^*AX}$, $(t \in I)$.

$$i.e., \begin{bmatrix} 2t^3-1-t^6 & 0 & 0 \\ 0 & 2t^3-1-t^6 & 0 \\ 0 & 0 & 2t^3-1-t^6 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{t}(2t^3-1-t^6) & 0 & 0 \\ 0 & t^2(2t^3-1-t^6) & 0 \\ 0 & 0 & t^2(2t^3-1-t^6) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad (t \in I).$$

This gives $x'_1 = \frac{1}{t}x_1$, $x'_2 = t^2x_2$, $x'_3 = t^2x_3$, $(t \in I)$, if $2t^3 - 1 - t^6 \neq 0$.

Hence, on integration, one obtains

$$x_1 = C_1 t, \quad x_2 = C_2 \exp\left(\frac{1}{3}t^3\right), \quad x_3 = C_3 \exp\left(\frac{1}{3}t^3\right), \quad (t \in I), \tag{ii}$$

where C_1, C_2, C_3 $(\in \mathbb{R})$ are the parameters of integration.

If $2t^3 - 1 - t^6 = 0$, then $t = 1$; fortunately, the equations (ia) – (ib) – (ic) are satisfied by x_1, x_2, x_3, obtained in (ii), for all $t \in \mathbb{R}$, including $t = 1$.

4. APPROACH III : BY USING INTEGRATING FACTORS OBTAINABLE FROM THE ADJOINT SODES OF THE SODES (1.2)

Given the SODEs

$$x_i' = \sum_{j=1}^{n} a_{ij}x_j, \quad i = 1, 2, ..., n, \quad (t \in I), \tag{4.1}$$

where $a_{ij} : I \to \mathbf{C}$ $(i, j = 1, 2, ..., n)$, the aim is to find a set of functions $\lambda_i : I \to C$, $i = 1, 2, ..., n$, such that

$$\sum_{i=1}^{n} \lambda_i x_i' = \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i a_{ij} x_j, \tag{4.2}$$

becomes an exact ODE.

The set of functions $\lambda_1, \lambda_2, ..., \lambda_n$ for which (4.2) becomes an exact ODE, may be termed **a set of integrating factors** of the given SODEs (4.1).

As any homogeneous SODEs of the form (1.2) can be converted to SODEs of the form (4.1) by multiplying (1.2) by $L^* =$ adj L, it is enough to consider SODEs of the form (4.1).

Rewriting (4.2) as

$$\left(\sum_{i=1}^{n} \lambda_i x_i\right)' = \sum_{i=1}^{n} \lambda_i' x_i + \sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ji} \lambda_j, \tag{4.3}$$

it is noted that (4.2) will become an exact ODE if the right hand side of (4.3) is zero for all admissible $x_1, x_2, ..., x_n$, i.e., if

$$\lambda_i' + \sum_{j=1}^{n} a_{ji} \lambda_j = 0, \quad i = 1, 2, ..., n. \tag{4.4}$$

The SODEs (4.4) may be termed the **SODEs adjoint to the given SODEs (4.1).**

To each solution $\lambda_1, \lambda_2, ..., \lambda_n$ of the adjoint SODEs (4.4), one derives from (4.3) an algebraic relation of the form

$$\sum_{i=1}^{n} \lambda_i x_i = C, \quad (C \in \mathbf{C}). \tag{4.5}$$

Hence, corresponding to n linearly independent solutions of (4.4), n linear algebraic equations of the form (4.5) are obtained, which are linearly independent. The solutions of these n linearly independent algebraic equations are the required solutions of the given SODEs (4.1).

Example 4 :

$$x'_1 = -2tx_2 + 2x_3 \tag{ia}$$

$$x'_2 = \frac{1}{1-t}x_1 + \frac{t^2-2}{1-t}x_2 + \frac{1+t}{t}x_3, \tag{ib}$$

$$x'_3 = \frac{t}{1-t}x_1 + \frac{t\left(t^2-2\right)}{1-t}x_2 + \frac{1+t+t^2}{t}x_3, \tag{ic}$$

for $t \in]0, 1[= \mathrm{I}$ (say), $x_i : \mathrm{I} \to \mathbb{R}$, $i = 1, 2, 3$.

The aim is to find functions $\lambda_i : \mathrm{I} \to \mathbb{R}$, $i = 1, 2, 3$, so that

$$(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)' = g(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3), \quad (t \in I), \tag{ii}$$

for some function $g : I \to \mathbb{R}$.

Using (ia), (ib), (ic) in (ii) one obtains

$$\lambda_1\left[-2tx_2 + 2x_3\right] + \lambda_2\left[\frac{1}{1-t}x_1 + \frac{t^2-2}{1-t}x_2 + \frac{1+t}{t}x_3\right]$$

$$+ \lambda_3\left[\frac{t}{1-t}x_1 + \frac{t\left(t^2-2\right)}{1-t}x_2 + \frac{1+t+t^2}{t}x_3\right]$$

$$+ \lambda'_1 x_1 + \lambda'_2 x_2 + \lambda'_3 x_3 = g(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3), \quad (t \in I). \tag{iii}$$

Equating the coefficeints of x_1, x_2, x_3 from the two sides of (iii) one obtains

$$\lambda'_1 + \frac{1}{1-t}\lambda'_2 + \frac{t}{1-t}\lambda_3 = g\lambda_1, \quad (t \in I), \tag{iva}$$

$$\lambda'_2 - 2t\lambda_1 + \frac{t^2-2}{1-t}\lambda_2 + \frac{t\left(t^2-2\right)}{1-t}\lambda_3 = g\lambda_2, \quad (t \in I), \tag{ivb}$$

$$\lambda'_3 + 2\lambda_1 + \frac{1+t}{t}\lambda_2 + \frac{1+t+t^2}{t}\lambda_3 = g\lambda_3, \quad (t \in I). \tag{ivc}$$

Eliminating λ_1 between (ivb) and (ivc) one gets

$$\lambda_2' + t\lambda_3' + \left(\frac{t^2-2}{1-t} + 1 + t\right)\lambda_2 + \left\{\frac{t\left(t^2-2\right)}{1-t} + 1 + t + t^2\right\}\lambda_3 = g\left(\lambda_2 + t\lambda_3\right).$$

After simplification one obtains

$$\left(\lambda_2 + t\lambda_3\right)' - \frac{1}{1-t}\left(\lambda_2 + t\lambda_3\right) = g\left(\lambda_2 + t\lambda_3\right).$$

Hence, on integration, one gets

$$(\lambda_2 + t\lambda_3)(1-t) = A_1 \exp\left(\int_a^t g(u)\,du\right), \quad (t \in I), \tag{v}$$

where A_1 ($\in \mathbb{R}$) is the parameter of integration and $a(\in I)$ is chosen arbitrarily.

Eliminating λ_3 between (iva) and (ivb) one gets

$$(t^2-2)\lambda'_1 - \lambda'_2 + 2t\lambda_1 = g\{(t^2-2)\lambda_1 - \lambda_2\},$$

whence, on integration, one obtains

$$(t^2-2)\lambda_1 - \lambda_2 = A_2 \exp\left(\int_a^t g(u)\,du\right) \tag{vi}$$

where A_2 ($\in \mathbb{R}$) is the parameter of integration and a ($\in I$) is chosen arbitrarily.

Eliminating λ_2 between (iva) and (ivc) one gets

$$\frac{1+t}{t}\lambda_1' - \frac{1}{1-t}\lambda_3' - \frac{2}{1-t}\lambda_1 + \left\{\frac{1+t}{t}\frac{t}{1-t} - \frac{1+t+t^2}{t(1-t)}\right\}\lambda_3 = g\left\{\frac{1+t}{t}\lambda_1 - \frac{1}{1-t}\lambda_3\right\},$$

whence, on integration, one obtains

$$(1-t^2)\lambda_1 - t\lambda_3 = A_3 \exp\left(\int_\alpha^t g(u)\,du\right), \quad (t \in I), \tag{vii}$$

where A_3 ($\in$ **R**) is the parameter of integration and a ($\in$ I) is chosen arbitrarily.

Taking $A_1 = A_2 = 0$ in (v) and (vi) one obtains

$$\frac{\lambda_1}{t} = \frac{\lambda_2}{t\left(t^2-2\right)} = \frac{\lambda_3}{-\left(t^2-2\right)}.$$

It can be verified that t(ia) + $t(t^2 - 2)$ (ib) − $(t^2 - 2)$ (ic) leads to

$$x_1' + \left(t^2-2\right)x_2' + 2tx_2 + \left(\frac{2}{t} - t\right)x_3' - \left(\frac{2}{t^2}+1\right)x_3 = 0, \qquad (t \in I),$$

whence, on integration, one obtains

$$x_1 + \left(t^2-2\right)x_2 + \left(\frac{2}{t} - t\right)x_3 = C_1, \qquad (t \in I), \tag{viii}$$

where C_1 ($\in$ **R**) is the parameter of integration.

Taking $A_2 = A_3 = 0$ in (vi) and (vii) one gets

$$\frac{\lambda_1}{t} = \frac{\lambda_2}{t\left(t^2-2\right)} = \frac{\lambda_3}{1-t^2}.$$

It can be verified that $t(ia) + t(t^2 - 2)(ib) + (1 - t^2)$ (ic) leads to

$$\frac{x_1'}{1-t} + \frac{x_1}{(1-t)^2} + \frac{t^2-2}{1-t}x_2' + \frac{(1-t)2t + \left(t^2-2\right)}{(1-t)^2}x_2 + \left(\frac{1}{t}+1\right)x_3' - \frac{1}{t^2}x_3 = 0,$$

whence, on integration, one obtains

$$\frac{x_1}{1-t} + \frac{t^2-2}{1-t}x_2 + \frac{1+t}{t}x_3 = C_2, \qquad (t \in I), \tag{ix}$$

where C_2 ($\in$ **R**) is the parameter of integration.

Taking $A_1 = A_3 = 0$ in (v) and (vii) one gets

$$\frac{\lambda_1}{t} = \frac{\lambda_2}{-t\left(1-t^2\right)} = \frac{\lambda_3}{1-t^2}.$$

It can be verified that $t(ia) - t(1 - t^2(ib) + (1 - t^2)$ (ic) leads to

$$x_1' + (t^2 - 1)x_2' + 2tx_2 + \left(\frac{1}{t} - t\right)x_3' - \left(\frac{1}{t^2} + 1\right)x_3 = 0, \quad (t \in I),$$

which, on integration, gives

$$x_1 + (t^2 - 1)x_2 + \left(\frac{1}{t} - t\right)x_3 = C_3, \quad (t \in I), \tag{x}$$

where C_3 ($\in \mathbf{R}$) is the parameter of integration.

Solving (viii), (ix), (x) one obtains, for all $t \in I$,

$$x_1 = (C_1 - C_3)t^2 + 2C_3 - C_1,$$

$$x_2 = (C_3 - C_2) + C_2 t,$$

$$x_3 = (C_1 - C_2)t + C_3 t^2.$$

5. REMARKS

The necessary and sufficient condition for an individual equation of the homogeneous SODEs (1.2) ($m = n = 2$) to be convertible to an exact differential equation has been determined in Case I*a* of §2.

In the rest of this article, a suitable linear combination of the equations of the homogeneous SODEs (1.2) is sought which would be integrable. Such a suitable linear combination of the equations of the given SODEs (1.2) is previously used to be determined **by inspection.** Attempts have been made here to chalk out an **algorithm** for the determination of requisite number of such suitable linear combinations of the equations of the given homogeneous SODEs (1.2). Three approaches for finding such linear combinations have been presented in §§2, 3, 4. However, the approaches discussed here are unfortunately not capable of covering all possible homogeneous SODEs. One such example will be discussed in a subsequent paper.

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Jour. Pure Math., Vol. 28, 2013, pp. 19-42

A UNIFIED THEORY OF OPEN FUNCTIONS IN BITOPOLOGICAL SPACES

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ABSTRACT : By using almost M-open functions from an m-space into an m-space, we establish the unified charactrizations for several generalized forms of open functions between bitopological spaces.

Key words nd phrases : m-structure, m-open set, almost-M-open function, bitopological space.
AMS Subject Classification (2010): 54A05, 54C10, 54E55.

1. INTRODUCTION

Semi-open sets, preopen sets, α-open sets, β-open sets and b-open sets play an important role in the researching of generalizations of open functions in topological spaces and bitopological spaces. By using these sets, several authors introduced and studied various types of modifications of open functions in topological spaces and bitopological spaces. Maheshwari and Prasad [14] and Bose [1] introduced the concepts of semi-open sets and semi-open functions in bitopological spaces. Jelic' [4], [6], Kar and Bhattacharyya [7] and Khedr et al. [8] introduced and studied the concepts of preopen sets and preopen functions in bitopological spaces. The notions of α-open sets and α-open functions in bitopological spaces were studied in [5], [18] and [9].

Recently, in [22] and [23] the present authors introduced the notions of minimal structures, m-spaces and m-continuity. Quite recently, in [19] and [20], they have introduced the notion of m-open functions in bitopological spaces.

In the present paper, by using almost M-openness due to Mocanu [17], we obtain unified characterizations for sevaral generalizations of open functions between bitopological spaces. The main results of this paper are Theorems 5.1, 5.2 and 5.3.

2. PRELIMINARIES

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. Throughout the present paper, (X, τ) and (Y, σ) always denote topological spaces and (X, τ_1, τ_2) and (Y, σ_1, σ_2) denote bitopological spaces. The closure of A and the interior of A with respect to τ_i are denoted by $i\mathrm{Cl}(A)$ and $i\mathrm{Int}(A)$, respectively, for $i = 1, 2$.

Definition 2.1 : A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (or briefly *m-structure*) [22], [23] on X if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X) (or briefly (X, m)), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space.* Each member of m_X is said to be *m_X-open* (or briefly *m-open*) and the complement of an m_X-open set is said to be *m_X-closed* (or briefly *m-closed*).

Definition 2.2 : Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [16] as follows :

(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F,\ X - F \in m_X\}$,

(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A,\ U \in m_X\}$.

Lemma 2.1 : (Maki et al. [16]). *Let (X, m_X) be an m-space. For subsets A and B of X, the following properties hold:*

(1) $\mathrm{mCl}(X - A) = X - \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X - A) = X - \mathrm{mCl}(A)$,

(2) *If* $(X - A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,

(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,

(4) *If* $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,

(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,

(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Lemma 2.2.: (Popa and Noiri [22]). *Let (X, m_X) be an m-space and A a subset of X. Then* $x \in \mathrm{mCl}(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every* $U \in m_X$ *containing* x.

Definition 2.3 : A minimal structure m_X on a nonempty set X is said to have *property* $\mathcal{B}$ [16] if the union of any family of subsets belonging to m_X belongs to m_X.

Lemma 2.3 (Popa and Noiri [24]. *Let* (X, m_X) *be an m-space and* m_X *satisfy property* $\mathcal{B}$.*Then for a subset* A *of* X, *the following properties hold:*

(1) $A \in m_X$ *if and only if* mInt$(A) = A$,

(2) A *is* m_X*-closed if and only if* mCl$(A) = A$,

(3) Int$(A) \in m_X$ *and* mCl(A) *is* m_X*-closed.*

3. ALMOST *M*-OPEN FUNCTIONS

Definition 3.1 : A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *almost M-open* at $x \in X$ [17] if for each m_X-open set U containing x, there exists $V \in m_Y$ containing $f(x)$ such that $V \subset f(U)$. If f is almost M-open at each point $x \in X$, then f is said to be *almost M-open.*

Theorem 3.1 *A function* $f : (X, m_X) \to (Y, m_Y)$ *is almost M-open at* $x \in X$ *if and only if for each* m_X*-open set* U *containing* x, x f^{-1}(mInt($f(U)$)).

Proof. *Necessity.* Let U be an m_X-open set containing x. Then, there exists $V \in m_Y$ such that $f(x) \in V \subset f(U)$ and hence $f(x) \in$ mInt($f(U)$). Therefore, we obtain that $x \in f^{-1}$(mInt($f(U)$)).

Sufficiency. Suppose that $x \in f^{-1}$(mInt($f(U)$)) for each m_X-open set U containing x. Then $f(x) \in$ mInt($f(U)$). Hence there exists $V \in m_Y$ containing $f(x)$ such that $V \subset f(U)$. Therefore, f is almost M-open at x.

Theorem 3.2 *A function* $f : (X, m_X) \to (Y, m_Y)$ *is almost M-open if and only if* mInt($f(U)$) $= f(U)$ *for each* m_X*-open set* U *of* X.

Proof. *Necessity.* Let U be an m_X-open set of X and $x \in U$. Then, by Theorem 3.1, we have $x \in f^{-1}$(mInt($f(U)$)) and hence $f(x) \in$ mInt($f(U)$). Therefore, $f(U) \subset$ mInt($f(U)$) and by Lemma 3.1 $f(U) =$ mInt($f(U)$).

Sufficiency. Let $x \in X$ and U be an m_X-open set of X containing x. Then we have $f(x) \in f(U) =$ mInt($f(U)$). Therefore $x \in f^{-1}$(mInt($f(U)$)). By Theorem 3.1, f is almost M-open at

each point $x \in X$ and hence f is almost M-open.

Theorem 3.3 *For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*

(1) *f is almost M-open at x;*

(2) *$x \in \mathrm{mInt}(A)$ implies $x \in f^{-1}(\mathrm{mInt}(f(A)))$ for each $A \in \mathcal{P}(X)$;*

(3) *$x \in \mathrm{mInt}\,(f^{-1}(B))$ implies $x \in f^{-1}(\mathrm{mInt}(B))$ for each $B \in \mathcal{P}(Y)$;*

(4) *$x \in f^{-1}(\mathrm{mCl}(B))$ implies $x \in \mathrm{mCl}(f^{-1}(B))$ for each $B \in \mathcal{P}(Y)$.*

Proof. (1) $\Rightarrow$ (2) : Let $A \in \mathcal{P}(X)$ and $x \in \mathrm{mInt}(A)$. Then, there exists an m_X-open set U such that $x \in U \subset A$ and hence $f(x) \in f(U) \subset f(A)$. Since f is almost M-open at x, by Theorem 3.1 $x \in f^{-1}(\mathrm{mInt}(f(U))) \subset f^{-1}(\mathrm{mInt}(f(A)))$.

(2) $\Rightarrow$ (3) : Let $B \in \mathcal{P}(Y)$ and $x \in \mathrm{mInt}(f^{-1}(B))$. Then $x \in f^{-1}(\mathrm{mInt}(f(f^{-1}(B)))) \subset f^{-1}(\mathrm{mInt}(B))$. Therefore, we have $x \in f^{-1}(\mathrm{mInt}(B))$.

(3) $\Rightarrow$ (4) : Let $B \in \mathcal{P}(Y)$ and $x \notin \mathrm{mCl}(f^{-1}(B))$. Then $x \in X - \mathrm{mCl}(f^{-1}(B)) = \mathrm{mInt}(X - f^{-1}(B)) = \mathrm{mInt}(f^{-1}(Y - B))$. By (3), we have $x \in f^{-1}(\mathrm{mInt}(Y - B)) = X - f^{-1}(\mathrm{mCl}(B)$. Therefore, $x \notin f^{-1}(\mathrm{mCl}(B))$.

(4) $\Rightarrow$ (1) : Let U be an m_X-open set of X containing x and $B = Y - f(U)$. Since $\mathrm{mCl}(f^{-1}(B)) = \mathrm{mCl}(f^{-1}(Y - f(U))) = \mathrm{mCl}(X - f^{-1}(f(U))) \subset X - \mathrm{mInt}(U) = X - U$ and $x \in U$, we obtain that $x \notin \mathrm{mCl}(f^{-1}(B))$. By (4), we have $x \notin f^{-1}(\mathrm{mCl}(B)) = f^{-1}(\mathrm{mCl}(Y - f(U))) = X - f^{-1}(\mathrm{mInt}\, f(U)))$. Therefore, we have $x \in f^{-1}(\mathrm{mInt}(f(U)))$. By Theorem 3.1, f is almost M-open at x.

For a function $f : (X, m_X) \to (Y, m_Y)$, we denote

$$D_{AMO}(f) = \{x \in X : f \text{ is not almost } M\text{-open at } x\}.$$

Theorem 3.4 *For a function $f : (X, m_X) \to (Y, m_Y)$, the following equalities hold:*

$$D_{AMO}(f) = \cup_{U \in m_x}\{U - f^{-1}(mInt(f(U)))\}$$

$$= \cup_{A \in \mathcal{P}(x)} \{\mathrm{mInt}(A) - f^{-1}(\mathrm{mInt}(f(A)))\}$$

$= \cup_{B\in \mathcal{P}(y)} \{\mathrm{mInt}(f^{-1}(B)) - f^{-1}(\mathrm{mInt}(B))\}$

$= \cup_{B\in \mathcal{P}(y)} \{f^{-1}(\mathrm{mCl}(B)) - \mathrm{mCl}(f^{-1}(B))\}$.

Proof. For the first equation, let $x \in D_{AMO}(f)$. Then, by Theorem 3.1, there exists an m_X-open set U_0 containing x such that $x \notin f^{-1}(\mathrm{mInt}(f(U_0)))$. Hence $x \in U_0 \cap (X - f^{-1}(\mathrm{Int}(f(U_0))))$ $= U_0 - f^{-1}(\mathrm{mInt}(f(U_0))) \subset \cup_{U\in m_X}\left\{U - f^{-1}\left(\mathrm{mInt}\left(f(U)\right)\right)\right\}$.

Conversely, let $x \in \cup_{U\in m_X}\left\{U - f^{-1}\left(\mathrm{mInt}\left(f(U)\right)\right)\right\}$. Then there exists $U_0 \in m_X$ such that $x \in U_0 - f^{-1}(\mathrm{mInt}(f(U_0)))$. Therefore, by Theorem 3.1 $x \in D_{AMO}(f)$.

For the second equation let $x \in D_{AMO}(f)$. Then, by Theorem 3.3 there exists $A_1 \in \mathcal{P}(X)$ such that $x \in \mathrm{mInt}(A_1)$ and $x \notin f^{-1}(\mathrm{mInt}(f(A_1)))$. Therefore, $x \in \mathrm{mInt}(A_1) - f^{-1}(\mathrm{mInt}(f(A_1)))$ $\subset \cup_{A\in \mathcal{P}(X)}\{\mathrm{mInt}(A) - f^{-1}(\mathrm{mInt}(f(A)))\}$.

Conversely, $x \in \cup_{A\in \mathcal{P}(X)}\{\mathrm{mInt}(A) - f^{-1}(\mathrm{mInt}(f(A)))\}$. Then there exists $A_1 \in \mathcal{P}(X)$ such that $x \in \mathrm{mInt}(A_1) - f^{-1}(\mathrm{mInt}(f(A_1)))$. By Theorem 3.3, $x \in D_{AMO}(f)$. The other equations are similarly proved.

Theorem 3.5 *For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*

(1) *f is almost M-open;*

(2) *$f(U) = \mathrm{mInt}(f(U))$ for each $U \in m_X$;*

(3) *$f(\mathrm{mInt}(A)) \subset \mathrm{mInt}(f(A))$ for each $A \in \mathcal{P}(X)$;*

(4) *$\mathrm{mInt}(f^{-1}(B)) \subset f^{-1}(\mathrm{mInt}(B))$ for each $B \in \mathcal{P}(Y)$;*

(5) *$f^{-1}(\mathrm{mCl}(B)) \subset \mathrm{mCl}(f^{-1}(B))$ for each $B \in \mathcal{P}(Y)$;*

(6) *For each $W \in \mathcal{P}(Y)$ and each $F \in \mathcal{P}(X)$ such that $f^{-1}(W) \subset F = \mathrm{mCl}(F)$, there exists $H \in \mathcal{P}(Y)$ such that $W \subset H = \mathrm{mCl}(H)$ and $f^{-1}(H) \subset F$.*

Proof. The proof follows from Theorem 3.2, Lemma 3.1 of [17] and Theorems 5.1 and 5.2 of [17].

By putting $m_X = \tau$ in Definition 3.1, we obtain the following definition.

Definition 3.2 Let (X, τ) be a topological space and (Y, m_Y) an m-space. A function $f : (X, \tau) \to (Y, m_Y)$ is said to be *m-open* at $x \in X$ [20] if for each open set U containing x, there exists $V \in m_Y$ containing $f(x)$ such that $V \subset f(U)$. If f is m-open at each point $x \in X$, then f is said to be *m-open.*

Corollary 3.1 (Noiri and Popa [20]). *For a function $f : (X, \tau) \to (Y, m_Y)$, the following properties are equivalent:*

(1) *f is m-open at x;*

(2) $x \in f^{-1}(\mathrm{mInt}(f(U)))$ *for each open set U containing x;*

(3) *If $x \in \mathrm{Int}(A)$ for $A \in \mathcal{P}(X)$, then $x \in f^{-1}(\mathrm{mInt}(f(A)))$;*

(4) *$x \in \mathrm{Int}(f^{-1}(B))$ for $B \in \mathcal{P}(Y)$, then $x \in f^{-1}(\mathrm{mInt}(B))$;*

(5) *If $x \in f^{-1}(\mathrm{mCl}(B))$ for $B \in \mathcal{P}(Y)$, then $x \in \mathrm{Cl}(f^{-1}(B))$.*

For a function $f : (X, \tau) \to (Y, m_Y)$, we denote

$$D_{mO}(f) = \{x \in X : f \text{ is not } m\text{-open at } x\}.$$

Corollary 3.2 (Noiri and Popa [20]). *For a function $f : (X, \tau) \to (Y, m_Y)$, the following equalities hold:*

$$\begin{aligned} D_{mO}(f) &= \cup_{U\in\tau}\{U - f^{-1}(\mathrm{mInt}(f(U)))\} \\ &= \cup_{A\in\mathcal{P}(x)}\{\mathrm{Int}(A) - f^{-1}(\mathrm{mInt}(f(A)))\} \\ &= \cup_{B\in\mathcal{P}(y)}\{\mathrm{Int}(f^{-1}(B)) - f^{-1}(\mathrm{mInt}(B))\} \\ &= \cup_{B\in\mathcal{P}(y)}\{f^{-1}(\mathrm{mCl}(B)) - \mathrm{Cl}(f^{-1}(B))\}. \end{aligned}$$

Corollary 3.3 (Noiri and Popa [19]). *For a function $f : (X, \tau) \to (Y, m_Y)$, the following properties are equivalent:*

(1) *f is m-open;*

(2) *$f(U) = \mathrm{mInt}(f(U))$ for each open set U of X;*

(3) *$f(\mathrm{Int}(A)) \subset \mathrm{mInt}(f(A)))$ for each $A \in \mathcal{P}(X)$;*

(4) $\mathrm{Int}(f^{-1}(B)) \subset f^{-1}(\mathrm{mInt}(B))$ *for each* $B \in \mathcal{P}(Y)$;

(5) $f^{-1}(\mathrm{mCl}(B)) \subset \mathrm{Cl}(f^{-1}(B))$ *for each* $B \in \mathcal{P}(Y)$.

Corollary 3.4 (Noiri and Popa [19]). *For a function* $f : (X, \tau) \to (Y, m_Y)$, *where* m_Y *has property* B, *the following properties are equivalent:*

(1) f *is m-open;*

(2) *For each* $S \in \mathcal{P}(Y)$ *and each closed set* F *of* X *containing* $f^{-1}(S)$, *there exists an m-closed set* H *of* Y *containing* S *such that* $f^{-1}(H) \subset F$.

By putting $m_Y = \sigma$ in Definition 3.1, we obtain the following definition.

Definition 3.3 Let (X, m_X) be an m-space and (Y, σ) a topological space. A function $f : (X, m_X) \to (Y, \sigma)$ is said to be *quasi m-open* at $x \in X$ if for each m_X-open set U containing x, there exists an open set V of Y containing $f(x)$ such that $V \subset f(U)$. If f is quasi m-open at each point $x \in X$, then f is said to be *quasi m-open.*

Corollary 3.5 *For a function* $f : (X, m_X) \to (Y, \sigma)$, *the following properties are equivalent:*

(1) f *is quasi m-open at* x;

(2) $x \in f^{-1}(\mathrm{Int}(f(U)))$ *for each* m_X*-open set* U *containing* x;

(3) $x \in \mathrm{mInt}(A)$ *implies* $x \in f^{-1}(\mathrm{Int}(f(A)))$ *for each* $A \in \mathcal{P}(X)$;

(4) $x \in \mathrm{mInt}(f^{-1}(B))$ *implies* $x \in f^{-1}(\mathrm{Int}(B))$ *for each* $B \in \mathcal{P}(Y)$;

(5) $x \in f^{-1}(\mathrm{Cl}(B))$ *implies* $x \in \mathrm{mCl}(f^{-1}(B))$ *for each* $B \in \mathcal{P}(Y)$.

Proof. This follows immediately from Theorems 3.1 and 3.3.

For a function $f : (X, m_X) \to (Y, \sigma)$, we denote

$$D_{qmO}(f) = \{x \in X : f \text{ is not quasi } m\text{-open at } x\}.$$

Corollary 3.6 *For a function* $f : (X, m_X) \to (Y, \sigma)$, *the following qualities hold:*

$$D_{qmO}(f) = \cup_{U \in m_X}\left\{U - f^{-1}\left(Int\left(f(U)\right)\right)\right\}$$

$$= \cup_{A \in \mathcal{P}(x)}\{\mathrm{mInt}(A) - f^{-1}(\mathrm{Int}(f(A)))\}$$

$$= \cup_{B \in \mathcal{P}(y)} \{ \mathrm{mInt}(f^{-1}(B)) - f^{-1}(\mathrm{Int}(B)) \}$$
$$= \cup_{B \in \mathcal{P}(y)} \{ f^{-1}(\mathrm{Cl}(B)) - \mathrm{mCl}(f^{-1}(B)) \}.$$

Corollary 3.7 *For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is quasi m-open;*

(2) *$f(U)$ is open in Y for each $U \in m_X$;*

(3) *$f(\mathrm{mInt}(A)) \subset \mathrm{Int}(f(A)))$ for each $A \in \mathcal{P}(X)$;*

(4) *$\mathrm{mInt}(f^{-1}(B)) \subset f^{-1}(\mathrm{Int}(B))$ for each $B \in \mathcal{P}(Y)$;*

(5) *$f^{-1}(\mathrm{Cl}(B)) \subset \mathrm{mCl}(f^{-1}(B))$ for each $B \in \mathcal{P}(Y)$.*

Corollary 3.8 *For a function $f : (X, m_X) \to (Y, \sigma)$, where m_X has property $\mathcal{B}$, the following properties are equivalent:*

(1) *f is m-open;*

(2) *For each $W \in \mathcal{P}(Y)$ and each m-closed set F of X such that $f^{-1}(W)$, there exists a closed set H of Y such that $W \subset H$ and $f^{-1}(H) \subset F$.*

4. MINIMAL STRUCTURES ON BITOPOLOGICAL SPACES

We shall recall some definitions of weak forms of open sets in a bitopological space.

A. (i, j)MO(X)

Definition 4.1 A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) *(i, j)-semi-open* [14] if $A \subset j\mathrm{Cl}(i\mathrm{Int}(A))$, where $i \neq j$, $i, j = 1, 2$,

(2) *(i, j)-preopen* [4] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(A))$, where $i \neq j$, $i, j = 1, 2$,

(3) *(i, j)-α-open* [5] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(i\mathrm{Int}(A)))$, where $i \neq j$, $i, j = 1, 2$,

(4) *(i, j)-b-open* [29] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(A)) \cup j\mathrm{Cl}(i\mathrm{Int}(A))$, where $i \neq j$, $i, j = 1, 2$,

(5) *(i, j)-β-open* or *(i, j)-semi-preopen* [8] if there exists an (i, j)-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j$, $i, j = 1, 2$.

The family of all (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-α-open, (i, j)-b-open, (i, j)-semi-preopen) sets of a bitopological space (X, τ_1, τ_2) is denoted by $(i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{BO}(X)$, $(i, j)\mathrm{SPO}(X)$).

Remark 4.1 Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. Then $(i, j)\mathrm{SO}(X)$, $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{BO}(X)$ and $(i, j)\mathrm{SPO}(X)$ are all m-structures on X. Hence, if $m_{ij}=(i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{BO}(X)$ and $(i, j)\mathrm{SPO}(X)$), then (1) $\mathrm{m_{ij}Cl}(A) = (i, j)\mathrm{sCl}(A)$ (resp. $(i, j)\mathrm{pCl}(A)$, $(i, j)\alpha\mathrm{Cl}(A)$, $(i, j)\mathrm{bCl}(A)$, $(i, j)\mathrm{spCl}(A)$), (2) $\mathrm{m_{ij}Int}(A) = (i, j)\mathrm{sInt}(A)$ (resp. $(i, j)\mathrm{pInt}(A)$, $(i, j)\alpha\mathrm{Int}(A)$, $(i, j)\mathrm{bInt}(A)$, $(i, j)\mathrm{spInt}(A)$).

Remark 4.2 Let (X, τ_1, τ_2) be a bitopological space.

(1) Let $m_{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\alpha(X)$). Then, by Lemma 2.1 we obtain the result established in Theorem 13 of [14] and Theorem 1.13 of [11] (resp. Theorem 3.6 of [18]).

(2) Let $m_{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$). Then, by Lemma 2.2 we obtain the result established in Theorem 1.15 of [11] (resp. Theorem 3.5 of [8], Theorem 3.5 of [18], Theorem 3.5 of [8]).

Remark 4.3 Let (X, τ_1, τ_2) be a bitopological space.

(1) It follows from Theorem 2 of [14] (resp. Theorem 4.2 of [7] or Theorem 3.2 of [8], Theorem 3.2 of [18], [29], Theorem 3.2 of [8] that $(i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{BO}(X)$, $(i, j)\mathrm{SPO}(X)$) is an m-structure on X satisfying property $\mathcal{B}$.

(2) Let $m_{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$). Then, by Lemma 2.3 we obtain the result established in Theorem 1.13 of [11] (resp. Theorem 3.5 of [8], Theorem 3.6 of [18], Theorem 3.6 of [8]).

B. QMO(X)

We shall recall some definitions of variations of quasi-open sets in bitopological spaces.

Definition 4.2 A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) *quasi-open* [3], [28] or $\tau_1\tau_2$-*open* [12] if $A = B \cup C$, where $B \in \tau_1$ and $C \in \tau_2$,

(2) *quasi-semi-open* [10], [15] if $A = B \cup C$, where B $\in$ SO(X, τ_1) and $C \in$ SO(X, τ_2),

(3) *quasi-preopen* [21] if $A = B \cup C$, where $B \in$ PO(X, τ_1) and $C \in$ PO(X, τ_2),

(4) *quasi-α-open* [30] if $A = B \cup C$, where $B \in \alpha$ (X, τ_1) and $C \in \alpha(X, \tau_2)$,

(5) *quasi-b-open* if $A = B \cup C$, where $B \in$ BO(X, τ_1) and $C \in$ BO(X, τ_2),

(6) *quasi-semi-preopen* [31] if $A = B \cup C$, where $B \in$ SPO(X, τ_1) and $C \in$ SPO(X, τ_2).

The family of all quasi-open (resp. quasi-semi-open, quasi-preopen, quasi-semi-preopen, quasi-α-open, quasi-b-open) sets of (X, τ_1, τ_2) is denoted by QO(X) or $\tau_1\tau_2$O(X) (resp. QSO(X), QPO(X), QSPO(X), Qα(X), QBO(X)).

The complement of a $\tau_1\tau_2$-open set is said to be $\tau_1\tau_2$-closed. The intersection of all $\tau_1\tau_2$-closed sets containing a subset A of X is called the $\tau_1\tau_2$-closure of A and is denoted by $\tau_1\tau_2$Cl(A). The union of all $\tau_1\tau_2$-open sets contained in a subset A of X is called the $\tau_1\tau_2$-interior of A and is denoted by $\tau_1\tau_2$Int(A).

Definition 4.3 Let (X, τ_1, τ_2) be a bitopological space and m_X^1 (resp. m_X^2) an m-structure on the topological space (X, τ_1) (resp. (X, τ_2)). The family

$$gm_X = \{A \subset X : A = B \cup C, \text{ where } B \in m_X^1 \text{ and } C \in m_X^2\}$$

is an m-structure on X which is called a *quasi m-structure* on X [2]. Each member of qm_X is said to be *quasi-m_X-open* (or briefly quasi-m-open). The complement of a quasi-m_X-open set is said to be *quasi-m_X-closed* (or briefly quasi-m-closed).

Remark 4.4 Let (X, τ_1, τ_2) be a bitopological space.

(1) If m_X^1 and m_X^2 have property $\mathcal{B}$, then qm_X has property $\mathcal{B}$.

(2) If $m_X^1 = \tau_1$ and $m_X^2 = \tau_2$ (resp. SO(X, τ_1) and SO(X, τ_2), PO(X, τ_1) and PO(X, τ_2), SPO(X, τ_1) and SPO(X, τ_2), BO(X, τ_1) and BO(X, τ_2), $\alpha(X, \tau_1)$ and $\alpha(X, \tau_2)$), then qm_X = QO(X) (resp. QSO(X), QPO(X), QSPO(X), QBO(X), Qα(X)).

(3) Since $SO(X, \tau_i)$, $PO(X, \tau_i)$, $SPO(X, \tau_i)$, $BO(X, \tau_i)$ and $\alpha(X, \tau_i)$ have property $\mathcal{B}$ for $i = 1, 2$, $QSO(X)$, $QPO(X)$, $QSPO(X)$, $QBO(X)$ and $Q\alpha(X)$ have property $\mathcal{B}$.

Definition 4.4 Let (X, τ_1, τ_2) be a bitopological space. For a subset A of X, the *quasi m_X-closure* of A and the *quasi m_X-interior* of A are defined as follows:

(1) $\mathrm{qmCl}(A) = \cap\{F : A \subset F, X - F \in qm_X\}$,

(2) $\mathrm{qmInt}(A) = \cup\{U : U \subset A, U \in qm_X\}$.

Remark 4.5 Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. If $qm_X = QO(X)$ (resp. $QSO(X)$, $QPO(X)$, $QSPO(X)$, $QBO(X)$, $Q\alpha(X)$), then we have

(1) $\mathrm{qmCl}(A) = \mathrm{qCl}(A)$ (resp. $\mathrm{qsCl}(A)$, $\mathrm{qpCl}(A)$, $\mathrm{qspCl}(A)$, $\mathrm{qbCl}(A)$, $\mathrm{q\alpha Cl}(A)$),

(2) $\mathrm{qmInt}(A) = \mathrm{qInt}(A)$ (resp. $\mathrm{qsInt}(A)$, $\mathrm{qpInt}(A)$, $\mathrm{qspInt}(A)$, $\mathrm{qbInt}(A)$, $\mathrm{q\alpha Int}(A)$).

C. $(1, 2)^*MO(X)$

Definition 4.5 A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) $(1, 2)^*$-*semi-open* [26], [27] if $A \subset \tau_1\tau_2\mathrm{Cl}(\tau_1\tau_2\mathrm{Int}(A))$,

(2) $(1, 2)^*$-*preopen* [26], [27] if $A \subset \tau_1\tau_2\mathrm{Int}(\tau_1\tau_2\mathrm{Cl}(A))$,

(3) $(1, 2)^*$-α-*open* [26], [27] if $A \subset \tau_1\tau_2\mathrm{Int}(\tau_1\tau_2\mathrm{Cl}(\tau_1\tau_2\mathrm{Int}(A)))$,

(4) $(1, 2)^*$-*semi-preopen* if $A \subset \tau_1\tau_2\mathrm{Cl}(\tau_1\tau_2\mathrm{Int}(\tau_1\tau_2\mathrm{Cl}(A)))$.

The family of all $(1, 2)^*$-semi-open (resp. $(1, 2)^*$-preopen, $(1, 2)^*$-α-open, $(1, 2)^*$-semi-preopen) sets is denoted by $(1, 2)^*SO(X)$ (resp. $(1, 2)^*PO(X)$, $(1, 2)^*\alpha(X)$, $(1, 2)^*SPO(X)$).

The complement of a $(1, 2)^*$-semi-open (resp. $(1, 2)^*$-preopen, $(1, 2)^*$-α-open, $(1, 2)^*$-semi-preopen) set is said to be $(1, 2)^*$-semi-closed (resp. $(1, 2)^*$-preclosed, $(1, 2)^*$-α-closed, $(1, 2)^*$-semi-preclosed).

The intersection of all $(1, 2)^*$-semi-closed (resp. $(1, 2)^*$-preclosed, $(1, 2)^*$-α-closed, $(1, 2)^*$-semi-preclosed) sets containing a subset A of X is called the $(1, 2)^*$-semi-closure (resp. $(1, 2)^*$-preclosure, $(1, 2)^*$-α-closure, $(1, 2)^*$-semi-preclosure) of A and is denoted by $(1, 2)^*\mathrm{sCl}(A)$ (resp. $(1, 2)^*\mathrm{pCl}(A)$, $(1, 2)^*\alpha\mathrm{Cl}(A)$, $(1, 2)^*\mathrm{spCl}(A)$).

The union of all (1, 2)*-semi-open (resp. (1, 2)*-preopen, (1, 2)*-α-open, (1, 2)*-semi-preopen) sets contained in A is called the (1, 2)*-semi-interior (resp. (1, 2)*-preinterior, (1, 2)*-α-interior, (1, 2)*-semi-preinterior) and is denoted by (1, 2)*sInt(A) (resp. (1, 2)*pInt(A), (1, 2)*αInt(A), (1, 2)*spInt(A)).

Remark 4.6 Let (X, τ_1, τ_2) be a bitopological space and A a subset of X.

(1) The families (1, 2)*SO(X), (1, 2)*PO(X), (1, 2)*α(X), and (1, 2)*SPO(X) are all m-structures with property $\mathcal{B}$.

(2) By (1) and Lemma 2.3, we have the following properties:

(i) A is (1, 2)*-semi-closed (resp. (1, 2)*-preclosed, (1, 2)*-α-closed, (1, 2)*-semi-preclosed) if and only if A = (1, 2)*sCl(A) (resp. A = (1, 2)*pCl(A), A = (1, 2)*αCl(A), A = (1, 2)*spCl(A)).

(ii) A is (1, 2)*-semi-open (resp. (1, 2)*-preopen, (1, 2)*-α-open, (1, 2)*-semi-preopen) if and only if A = (1, 2)*sInt(A) (resp. A = (1, 2)*pInt(A), A = (1, 2)*αInt(A), A = (1, 2)*spInt(A)).

(3) By Lemma 2.2 we obtain the result established in Proposition 2.2(ii) of [28].

(4) By Lemma 2.1, we obtain the relations between (1, 2)*sCl(A) (resp. (1, 2)*pCl(A), (1, 2)*αCl(A), (1, 2)*spCl(A)) and (1, 2)*sInt(A) (resp. (1, 2)*pInt(A), (1, 2)*αInt(A), (1, 2)*spInt(A)).

D. (1, 2)MO(X)

Definition 4.6 A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) (1, 2)-*semi-open* [12] if $A \subset \tau_1\tau_2\mathrm{Cl}(\tau_1\mathrm{Int}(A))$,

(2) (1, 2)-*preopen* [12] if $A \subset \tau_1\mathrm{Int}(\tau_1\tau_2\mathrm{Cl}(A))$,

(3) (1, 2)-α-*open* [12] if $A \subset \tau_1\mathrm{Int}(\tau_1\tau_2\mathrm{Cl}(\tau_1\mathrm{Int}(A)))$,

(4) (1, 2)-*semi-preopen* [13], [25] if A $\tau_1\tau_2\mathrm{Cl}(\tau_1\mathrm{Int}(\tau_1\tau_2\mathrm{Cl}(A)))$.

The collection of all (1, 2)-semi-open (resp. (1, 2)-preopen, (1, 2)-α-open, (1, 2)-semi-preopen) sets of X is denoted by (1, 2)SO(X) (resp. (1, 2)PO(X), (1, 2)α(X), (1, 2)SPO(X)).

The complement of a (1, 2)-semi-open (resp. (1, 2)-preopen, (1, 2)-α-open (1, 2)-semi-preopen) set of X is said to be (1, 2)-*semi-closed* (resp. (1, 2)-*preclosed,* (1, 2)-α-*closed,* (1, 2)-*semi-preclosed*).

The intersection of all (1, 2)-semi-closed (resp. (1, 2)-preclosed, (1, 2)-α-closed, (1, 2)-semi-preclosed) sets containg A of X is called the (1, 2)-*semi-closure* (resp. (1, 2)-*preclosure,* (1, 2)-α-*closure,* (1, 2)-*semi-preclosure)* of A and is denoted by (1, 2)sCl(A) (resp. (1, 2)pCl(A), (1, 2)αCl(A), (1, 2)spCl(A)).

The union of all (1, 2)-semi-open (resp. (1, 2)-preopen, (1, 2)-α-open, (1, 2)-semi-preopen) sets of X contained in A is called the (1, 2)-*semi-interior* (resp. (1, 2)-*preinterior,* (1, 2)-α-*interior,* (1, 2)-*semi-preinterior*) of A and is denoted by (1, 2)sInt(A) (resp. (1, 2)pInt(A), (1, 2)αInt(A), (1, 2)spInt(A)).

Remark 4.7 Let (X, τ_1, τ_2) be a bitopological space and A a subset of X.

(1) The families (1, 2)SO(X), (1, 2)PO(X), (1, 2)αO(X) and (1, 2)SPO(X) are all m-structures on X having property $\mathcal{B}$.

(2) By (1) and Lemma 2.3, we have the following properties:

(i) A subset A is (1, 2)-semi-closed (resp. (1, 2)-preclosed, (1, 2)-α-closed, (1, 2)-semi-preclosed) if and only if A = (1, 2)sCl(A) (resp. A = (1, 2)pCl(A), A = (1, 2)αCl(A), A = (1, 2)spCl(A)).

(ii) A is (1, 2)-semi-open (resp. (1, 2)-preopen, (1, 2)-α-open (1, 2)-semi-preopen) if and only if A = (1, 2)sInt(A) (resp. A = (1, 2)pInt(A), A = (1, 2)αInt(A), A = (1, 2)spInt(A)).

(3) By Lemma 2.2, we obtain the results established in Lemma 8 of [27].

(4) By Lemma 2.1 we obtain the relations between (1, 2)sCl(A) (resp. (1, 2)pCl(A), (1, 2)αCl(A), (1, 2)spCl(A)) and (1, 2)sInt(A) (resp. (1, 2)pInt(A), (1, 2)αInt(A), (1, 2)spInt(A)).

5. ALMOST *M*-OPEN FUNCTIONS IN BITOPOLOGICAL SPACES

5.1 Almost *M*-open functions

First, we recall some generalized forms of open functions between bitopological spaces.

Definition 5.1 A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-presemi-open* (resp. *(1, 2)-preopen* [13], *$\tau_1\tau_2$-open* [27] if $f(U)$ is (i, j)-semi-open (resp. (1, 2)-preopen, $\sigma_1\sigma_2$-open) in Y for each (i, j)-semi-open (resp. (1, 2)-preopen $\tau_1\tau_2$-open) set U of X.

Since $(i, j)\mathrm{SO}(Y)$, $(1, 2)\mathrm{PO}(Y)$ and $\sigma_1\sigma_2\mathrm{O}(Y)$ have property ß, it follows that a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-presemi-open (resp. (1, 2)-preopen, $\tau_1\tau_2$-open) if $f : (X, m(\tau_1, \tau_2)) \to (Y, m(\sigma_1, \sigma_2))$ is almost M-open, where $m(\tau_1, \tau_2) = (i, j)\mathrm{SO}(X)$ (resp. $(1, 2)\mathrm{PO}(X)$ and $\tau_1\tau_2\mathrm{O}(X)$) and $m(\sigma_1, \sigma_2)) = (i, j)\mathrm{SO}(Y)$ (resp. $(1, 2)\mathrm{PO}(Y)$ and $\sigma_1\sigma_2\mathrm{O}(Y)$). As shown by the above functions, some modifications of open functions between bitopological spaces are characterized by almost M-open functions between m-spaces with appropriate m-structures determined by the topologies.

Definition 5.2 Let (X, τ_1, τ_2) (resp. (Y, σ_1, σ_2)) be a bitopological space and $m(\tau_1, \tau_2)$ (resp. $m(\sigma_1, \sigma_2)$) denote an m-structure on X (resp. Y) determined by τ_1 and τ_2 (resp. σ_1 and σ_2). A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *almost M-open* at $x \in X$ (resp. on X) if $f : (X, m(\tau_1, \tau_2)) \to (Y, m(\sigma_1, \sigma_2))$ is almost M-open at $x \in X$ (resp on X).

Remark 5.1 Let (X, τ_1, τ_2) be a bitopological space. For minimal structures $m(\tau_1, \tau_2)$ determined by τ_1 and τ_2, for example, we have the following.

A. $(i, j)\mathrm{MO}(X) = (i, j)\mathrm{SO}(X)$, $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{BO}(X)$, $(i, j)\mathrm{SPO}(X)$).

B. $\mathrm{QMO}(X) = \mathrm{QO}(X)$, $\mathrm{QSO}(X)$, $\mathrm{QPO}(X)$, $\mathrm{Q}\alpha(X)$, $\mathrm{QBO}(X)$, $\mathrm{QSPO}(X)$.

C. $(1, 2)^*\mathrm{MO}(X) = (1, 2)^*\mathrm{SO}(X)$, $(1, 2)^*\mathrm{PO}(X)$, $(1, 2)^*\alpha(X)$, $(1, 2)^*\mathrm{SPO}(X)$.

D. $(1, 2)\mathrm{MO}(X) = (1, 2)\mathrm{SO}(X)$, $(1, 2)\mathrm{PO}(X)$, $(1, 2)\alpha(X)$, $(1, 2)\mathrm{SPO}(X)$.

By Definition 5.2 and Theorems 3.1 and 3.3, we obtain the following theorem.

Theorem 5.1 *Let* (X, τ_1, τ_2) *(resp.* (Y, σ_1, σ_2)*) be a bitopological space and* $m(\tau_1, \tau_2)$ *(resp.*

$m(\sigma_1, \sigma_2)$ a minimal structure determined by τ_1 and τ_2 (resp. σ_1 and σ_2). For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) *f is almost M-open at $x \in X$;*

(2) *$x \in f^{-1}(m(\sigma_1, \sigma_2)\mathrm{Int}(f(U)))$ for every $m(\tau_1, \tau_2)$-open set U containing x;*

(3) *$x \in m(\tau_1, \tau_2)\mathrm{Int}(A)$ implies $x \in f^{-1}(m(\sigma_1, \sigma_2)\mathrm{Int}(f(A)))$ for each $A \in \mathcal{P}(X)$;*

(4) *$x \in m(\tau_1, \tau_2)\mathrm{Int}(f^{-1}(B))$ implies $x \in f^{-1}(m(\sigma_1, \sigma_2)\mathrm{Int}(B))$ for each $B \in \mathcal{P}(Y)$;*

(5) *$x \in f^{-1}((\sigma_1, \sigma_2)\mathrm{Cl}(B))$ implies $x \in m(\tau_1, \tau_2)\mathrm{Cl}(f^{-1}(B))$ for each $B \in \mathcal{P}(Y)$.*

By Theorems 3.2 and 3.5, we obtain the following theorem:

Theorem 5.2 *Let (X, τ_1, τ_2) (resp. (Y, σ_1, σ_2)) be a bitopological space and $m(\tau_1, \tau_2)$ (resp. $m(\sigma_1, \sigma_2)$) a minimal structure determined by τ_1 and τ_2 (resp. σ_1 and σ_2). For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is almost M-open;*

(2) *$f(U) = m(\sigma_1, \sigma_2)\mathrm{Int}(f(U))$ for each $U \in m(\tau_1, \tau_2)$;*

(3) *$f(m(\tau_1, \tau_2)\mathrm{Int}(A)) \subset m(\sigma_1, \sigma_2)\mathrm{Int}(f(A))$ for each $A \in \mathcal{P}(X)$;*

(4) *$m(\tau_1, \tau_2)\mathrm{Int}(f^{-1}(B)) \subset f^{-1}(m(\sigma_1, \sigma_2)\mathrm{Int}(B))$ for each $B \in \mathcal{P}(Y)$;*

(5) *$f^{-1}(\mathrm{m}(\sigma_1, \sigma_2)\mathrm{Cl}(B)) \subset m(\tau_1, \tau_2)\mathrm{Cl}(f^{-1}(B))$ for each $B \in \mathcal{P}(Y)$.*

(6) *For each $W \in \mathcal{P}(Y)$ and each $F \in \mathcal{P}(X)$ such that $f^{-1}(W) \subset F = m(\tau_1, \tau_2)\mathrm{Cl}(F)$, there exists $H \in \mathcal{P}(Y)$ such that $W \subset H = m(\sigma_1, \sigma_2)\mathrm{Cl}(H)$ and $f^{-1}(H) \subset F$.*

Corollary 5.1 *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is (i, j)-presemi-open;*

(2) *$f(U) \in (i, j)\mathrm{SO}(Y)$ for each $U \in (i, j)\mathrm{SO}(X)$;*

(3) *$f((i, j)\mathrm{sInt}(A)) \subset (i, j)\mathrm{sInt}(f(A)))$ for each $A \in \mathcal{P}(X)$;*

(4) *$(i, j)\mathrm{sInt}(f^{-1}(B)) \subset f^{-1}((i, j)\mathrm{sInt}(B))$ for each $B \in \mathcal{P}(Y)$;*

(5) *$f^{-1}((i, j)\mathrm{sCl}(B)) \subset (i, j)\mathrm{sCl}(f^{-1}(B))$ for each $B \in \mathcal{P}(Y)$.*

(6) *For each* $W \in \mathcal{P}(Y)$ *and each (i, j)-semi-closed set F such that* $f^{-1}(W) \subset F$, *there exists an (i, j)-semi-closed set H of Y such that* $W \subset H$ and $f^{-1}(H) \subset F$.

Proof. By putting $m(\tau_1, \tau_2) = (i, j)\mathrm{SO}(X)$ and $m(\sigma_1, \sigma_2) = (i, j)\mathrm{SO}(Y)$, we obtain the proof from Theorem 5.2 and Lemma 2.3.

Corollary 5.2 *For a function* $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, *the following properties are equivalent:*

(1) *f is (1, 2) -preopen;*

(2) $f(U) \in (1, 2)\mathrm{PO}(Y)$ *for each* $U \in (1, 2)\mathrm{PO}(X)$;

(3) $f((1, 2)\mathrm{pInt}(A)) \subset (1, 2)\mathrm{pInt}(f(A)))$ *for each* $A \in \mathcal{P}(X)$;

(4) $(1, 2)\mathrm{pInt}(f^{-1}(B)) \subset f^{-1}((1, 2)\mathrm{pInt}(B))$ *for each* $B \in \mathcal{P}(Y)$;

(5) $f^{-1}((1, 2)\mathrm{pCl}(B)) \subset (1, 2)\mathrm{pCl}(f^{-1}(B))$ *for each* $B \in \mathcal{P}(Y)$.

(6) *For each* $W \in \mathcal{P}(Y)$ *and each (1, 2)-preclosed set F such that* $f^{-1}(W) \subset F$, *there exists a (1, 2)-preclosed set H of Y such that* $W \subset H$ *and* $f^{-1}(H) \subset F$.

Proof. By putting $m(\tau_1, \tau_2) = (1, 2)\mathrm{PO}(X)$ and $m(\sigma_1, \sigma_2) = (1, 2)\mathrm{PO}(Y)$, we obtain the proof from Theorem 5.2 and Lemma 2.3.

Corollary 5.3 *For a function* $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, *the following properties are equivalent:*

(1) *f is* $\tau_1\tau_2$*-open;*

(2) $f(U) \in \sigma_1\sigma_2\mathrm{O}(Y)$ *for each* $U \in \tau_1\tau_2\mathrm{O}(X)$;

(3) $f(\tau_1\tau_2\mathrm{Int}(A)) \subset \sigma_1\sigma_2\mathrm{Int}(f(A)))$ *for each* $A \in \mathcal{P}(X)$;

(4) $\tau_1\tau_2\mathrm{Int}(f^{-1}(B)) \subset f^{-1}(\sigma_1\sigma_2\mathrm{Int}(B))$ *for each* $B \in \mathcal{P}(Y)$;

(5) $f^{-1}(\sigma_1\sigma_2\mathrm{Cl}(B)) \subset \tau_1\tau_2\mathrm{Cl}(f^{-1}(B))$ *for each* $B \in \mathcal{P}(Y)$.

(6) *For each* $W \in \mathcal{P}(Y)$ *and each* $\tau_1\tau_2$*-closed set F such that* $f^{-1}(W) \subset F$, *there exists a* $\sigma_1\sigma_2$*-preclosed set H of Y such that* $W \subset H$ *and* $f^{-1}(H) \subset F$.

Proof. By putting $m(\tau_1, \tau_2) = \tau_1\tau_2\mathrm{O}(X)$ and $m(\sigma_1, \sigma_2) = \sigma_1\sigma_2\mathrm{O}(Y)$, we obtain the proof from Theorem 5.2 and Lemma 2.3.

For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, we denote

$D_{AMO}(f) = \{x \in X : f \text{ is not almost M-open at } x\}$.

Let $m(\tau_1, \tau_2)$ (resp. $m(\sigma_1, \sigma_2)$ be a minimal structure on X (resp. Y) determined by τ_1 and τ_2 (resp. σ_1 and σ_2). Then, by Theorem 3.4 we obtain the following theorem.

Theorem 5.3 *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following equalities hold:*

$$D_{AMO}(f) = \cup_{U \in m(\tau_1,\tau_2)}\left\{U - f^{-1}\left(m(\sigma_1,\sigma_2)Int\left(f(U)\right)\right)\right\}$$

$$= \cup_{A \in \mathcal{P}(x)}\{m(\tau_1, \tau_2)\mathrm{Int}(A) - f^{-1}(m(\sigma_1, \sigma_2)\mathrm{Int}(f(A)))\}$$

$$= \cup_{B \in \mathcal{P}(y)}\{m(\tau_1, \tau_2)\mathrm{Int}(f^{-1}(B)) - f^{-1}(m(\sigma_1, \sigma_2)\mathrm{Int}(B))\}$$

$$= \cup_{B \in \mathcal{P}(y)}\{f^{-1}(m(\sigma_1, \sigma_2)\mathrm{Cl}(B)) - m(\tau_1, \tau_2)\mathrm{Cl}(f^{-1}(B))\}$$

For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, we denote

$D_{PSO}(f) = \{x \in X : f \text{ is not } (i, j)\text{-presemi-open at } x\}$.

Then, by Theorem 5.3 we obtain the following corollary:

Corollary 5.4 *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following equalities hold:*

$$D_{PSO}(f) = \cup_{U \in (ij)\mathrm{SO}(X)}\{U - f^{-1}((i, j)\mathrm{sInt}(f(U)))\}$$

$$= \cup_{A \in \mathcal{P}(X)}\{(i, j)\mathrm{sInt}(A) - f^{-1}((i, j)\mathrm{sInt}(f(A)))\}$$

$$= \cup_{B \in \mathcal{P}(y)}\{(i, j)\mathrm{sInt}(f^{-1}(B)) - f^{-1}((i, j)\mathrm{sInt}(B))\}$$

$$= \cup_{B \in \mathcal{P}(y)}\{f^{-1}((i, j)\mathrm{sCl}(B) - (i, j)\mathrm{sCl}(f^{-1}(B))\}$$

For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, we denote

$$D_{(1,2)PO}(f) = \{x \in X : f \text{ is not } (1, 2)\text{-preopen at } x\}.$$

Then, by Theorem 5.3 we obtain the following corollary:

Corollary 5.5 *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following equalities hold:*

$$D_{(1,\ 2)\mathrm{PO}}(f) = \cup_{\mathrm{U} \in (1,\ 2)\mathrm{PO}(X)}\{U - f^{-1}((1, 2)\mathrm{pInt}(f(U)))\}$$

$$= \cup_{A \in \mathcal{P}(x)} \{(1, 2)\mathrm{pInt}(A) - f^{-1}((1, 2)\mathrm{pInt}(f(A)))\}$$

$$= \cup_{B \in \mathcal{P}(y)} \{(1, 2)\mathrm{pInt}(B)) - f^{-1}((1, 2)\mathrm{pInt}(f(B))\}$$

$$= \cup_{B \in \mathcal{P}(y)} \{f^{-1}((1, 2)\mathrm{pCl}(B)) - (1, 2)\mathrm{pCl}(f^{-1}(B))\}$$

For a function $f\ (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, we denote

$$D_{\tau_1\tau_2 O}(f) = \{x \in X : f \text{ is not } \tau_1\tau_2\text{-open at } x\}.$$

Then, by Theorem 5.3 we obtain the following corollary:

Corollary 5.6 *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following equalities hold:*

$$D_{\tau_1\tau_2 O}(f) = \cup_{U \in \tau_1\tau_2 O(x)} \left\{U - f^{-1}\left(\sigma_1\sigma_2 Int\left(f(U)\right)\right)\right\}$$

$$= \cup_{A \in \mathcal{P}(x)} \{\tau_1\tau_2\mathrm{Int}(A) - f^{-1}(\sigma_1\sigma_2\mathrm{Int}(f(A)))\}$$

$$= \cup_{B \in \mathcal{P}(y)} \{\tau_1\tau_2\mathrm{Int}(B)) - f^{-1}(\sigma_1\sigma_2\mathrm{Int}(f(B))\}$$

$$= \cup_{B \in \mathcal{P}(y)} \{f^{-1}(\sigma_1\sigma_2\mathrm{Cl}(B)) - \tau_1\tau_2\mathrm{Cl}(f^{-1}(B))\}$$

5.2 *(i, j)-m-open functions*

Definition 5.3 A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-semi-open* [1] (resp. *(i, j)-preopen* [7], *(i, j)-α-open* [9], *(i, j)-semi-preopen*) if for each τ_i-open set U of X, $f(U)$ is *(i, j)*-semi-open (resp. *(i, j)*-preopen, *(i, j)*-α-open, *(i, j)*-semi-preopen) in Y.

Definition 5.4 Let (Y, σ_1, σ_2) be a bitopological space and m_{ij} a minimal structure on Y determined by σ_1 and σ_2. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-m-open at* $x \in X$ [20] (resp. on X [19]) if $f : (X, \tau_i) \to (Y, m_{ij})$ is m-open at x (resp. on X).

Remark 5.2 (1) By Definition 5.4, it follows that a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is *(i, j)-m*-open at $x \in X$ if and only if for each τ_i-open set U containing x, there exists an m_{ij}-open set V of Y such that $f(x) \subset V$ and $V \subset f(U)$.

(2) If $m_{ij} = (i, j)\mathrm{SO}(Y)$ (resp. $(i, j)\mathrm{PO}(Y)$, $(i, j)\alpha(Y)$, $(i, j)\mathrm{SPO}(Y)$ and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is *(i, j)-m*-open at $x \in X$ (resp. on X), then f is *(i, j)*-semi-open (resp. *(i, j)*-preopen, *(i, j)*-α-open, *(i, j)*-semi-preopen) at x (resp. on X).

(3) Since $(i, j)SO(Y)$, $(i, j)PO(Y)$, $(i, j)\alpha(Y)$ and $(i, j)SPO(Y)$ are all m-structures on Y satisfying property $\mathcal{B}$, a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-α-open, (i, j)-semi-preopen) if and only if $f : (X, \tau_i) \to (Y, m_{ij})$ is m-open, where $m_{ij} = (i, j)SO(Y)$ (resp. $(i, j)PO(Y)$, $(i, j)\alpha(Y)$, $(i, j)SPO(Y)$).

By Theorems 5.1 and 5.2 of [20], we obtain the following theorem.

Theorem 5.4 *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is (i, j)-m-open at $x \in X$;*

(2) *$x \in f^{-1}(m_{ij}\mathrm{Int}(f(U)))$ for every τ_i-open set U containing x;*

(3) *If $x \in i\mathrm{Int}(A)$ for $A \in \mathcal{P}(X)$, then $x \in f^{-1}(m_{ij}\mathrm{Int}(f(A)))$;*

(4) *If $x \in i\mathrm{Int}(f^{-1}(B))$ for $B \in \mathcal{P}(Y)$, then $x \in f^{-1}(m_{ij}\mathrm{Int}(B))$;*

(5) *If $x \in f^{-1}(m_{ij}\mathrm{Cl}(B))$ for $B \in \mathcal{P}(Y)$, then $x \in i\mathrm{Cl}(f^{-1}(B))$.*

By Theorems 5.1 and 5.2 of [19], we obtain the following theorems.

Theorem 5.5 *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is (i, j)-m-open;*

(2) *$f(U) = m_{ij}\mathrm{Int}(f(U))$ for every open set $U \in \tau_i$;*

(3) *$f(i\mathrm{Int}(A)) \subset m_{ij}\mathrm{Int}(f(A))$ for every subset A of X;*

(4) *$i\mathrm{Int}(f^{-1}(B)) \subset f^{-1}(m_{ij}\mathrm{Int}(B))$ for every subset B of Y;*

(5) *$f^{-1}(m_{ij}\mathrm{Cl}(B)) \subset i\mathrm{Cl}(f^{-1}(B))$ for every subset B of Y.*

Theorem 5.6 *Let m_{ij} be a minimal structure with property $\mathcal{B}$ on Y determined by σ_1 and σ_2. Then, for a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is (i, j)-m-open;*

(2) *$f(U)$ is m_{ij}-open for every τ_i-open set U of X;*

(3) *for any subset S of Y and each τ_i-closed set F of X containing $f^{-1}(S)$, there exists an m_{ij}-closed set H of Y containing S such that $f^{-1}(H) \subset F$.*

Remark 5.3 If $m_{ij} = (i, j)\mathrm{SO}(Y)$ (resp. $(i, j)\mathrm{PO}(Y)$, $(i, j)\alpha(Y)$), then by Theorems 5.5 and 5.6, we obtain the characterizations of (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-α-open) functions established in Theorem 3.1 of [1] (resp. Theorems 7.3 and 7.4 of [7] and Proposition 3.1 of [6], Theorem 4.6 and Corollary 4.7 of [9]).

For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, we denote

$$D_{(i,\, j)\mathrm{mO}}(f) = \{x \in X : f \textit{ is not } (i, j)\textit{-}m\textit{-open at } x\}.$$

Theorem 5.7 *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where m_{ij} is a minimal structure on Y determined by σ_1 and σ_2, the following equalities hold:*

$$\begin{aligned} D_{(i,\, j)\mathrm{mO}}(f) &= \cup_{U \in \tau_1}\left\{U - f^{-1}\left(m_{ij}\mathrm{Int}\left(f(U)\right)\right)\right\} \\ &= \cup_{A \in \mathcal{P}(x)}\{i\mathrm{Int}(A) - f^{-1}(m_{ij}\mathrm{Int}(f(A)))\} \\ &= \cup_{B \in \mathcal{P}(y)}\{i\mathrm{Int}(f^{-1}(B)) - f^{-1}(m_{ij}\mathrm{Int}(B))\} \\ &= \cup_{B \in \mathcal{P}(y)}\{f^{-1}(m_{ij}\mathrm{Cl}(B)) - i\mathrm{Cl}(f^{-1}(B))\}. \end{aligned}$$

5.3 (i, j)-quasi-m-open functions

Definition 5.5 Let (X, τ_1, τ_2) be a bitopological space and m_{ij} a minimal structure on X determined by τ_1 and τ_2. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-quasi-m-open* at $x \in X$ (resp. on X) if $f : (X, m_{ij}) \to (Y, \sigma_i)$ is quasi-m-open at x (resp. on X).

Remark 5.4 (1) By Definition 5.5, it follows that a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-quasi-m-open at $x \in X$ if and only if for each (i, j)-m-open set U containing x, there exists a σ_i-open set V of Y such that $f(x) \subset V$ and $V \subset f(U)$.

(2) If $m_{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$) and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-quasi-m-open at $x \in X$ (resp. on X), then f is (i, j)-quasi-semi-open (resp. (i, j)-quasi-preopen, (i, j)-quasi-α-open, (i, j)-quasi-semi-preopen) at x (resp. on X).

By Corollaries 3.7 and 3.8, we obtain the following theorems.

Theorem 5.8 *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is (i, j)-quasi-m-open;*

(2) *$f(U)$ is σ_i-open for every $U \in m_{ij}$;*

(3) *$f(m_{ij}\mathrm{Int}(A)) \subset \sigma_i\mathrm{Int}(f(A))$ for every subset A of X;*

(4) *$m_{ij}\mathrm{Int}(f^{-1}(B)) \subset f^{-1}(\sigma_i\mathrm{Int}(B))$ for every subset B of Y;*

(5) *$f^{-1}(\sigma_i\mathrm{Cl}(B)) \subset m_{ij}\mathrm{Cl}(f^{-1}(B))$ for every subset B of Y.*

Theorem 5.9 *Let (X, τ_1, τ_2) be a bitopological space and m_{ij} a minimal structure with property $\mathcal{B}$ on X determined by τ_1 and τ_2. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is (i, j)-quasi-m-open;*

(2) *for any subset S of Y and each m_{ij}-closed set F of X containing $f^{-1}(S)$, there exists a σ_i-closed set H of Y containing S such that $f^{-1}(H) \subset F$.*

Remark 5.5 (1) By Theorem 5.8, we obtain Theorems 2.4 and 2.6 of [29].

(2) By Theorem 5.9, we obtain Theorem 2.5 of [29].

Theorem 5.10 *Let (X, τ_1, τ_2) be a bitopological space and m_{ij} a minimal structure on X determined by τ_1 and τ_2. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is (i, j)-quasi-m-open at $x \in X$;*

(2) *$x \in f^{-1}(\sigma_i\mathrm{Int}(f(U)))$ for every m_{ij}-open set U containing x;*

(3) *If $x \in m_{ij}\mathrm{Int}(A)$ for $A \in \mathcal{P}(X)$, then $x \in f^{-1}(\sigma_i\mathrm{Int}(f(A)))$;*

(4) *If $x \in m_{ij}\mathrm{Int}(f^{-1}(B))$ for $B \in \mathcal{P}(Y)$, then $x \in f^{-1}(\sigma_i\mathrm{Int}(B))$;*

(5) *If $x \in f^{-1}(\sigma_i\mathrm{Cl}(B))$ for $B \in \mathcal{P}(Y)$, then $x \in m_{ij}\mathrm{Cl}(f^{-1}(B))$.*

For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, we denote

$D_{(i,\, j)qmO}(f) = \{x \in X : f$ *is not (i, j)-quasi-m-open at* $x\}$.

Theorem 5.11 *Let (X, τ_1, τ_2) be a bitopological space and m_{ij} a minimal structure on X determined by* τ_1 and τ_2. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following equalities hold:*

$$
\begin{aligned}
D_{(i,\, j)\mathrm{qmO}}(f) &= \cup_{U \in m_{ij}} \left\{ U - f^{-1}\left(\sigma_i \operatorname{Int}\left(f(U)\right)\right) \right\} \\
&= \cup_{A \in \mathcal{P}(x)} \{ m_{ij}\operatorname{Int}(A) - f^{-1}(\sigma_i \operatorname{Int}(f(A))) \} \\
&= \cup_{B \in \mathcal{P}(y)} \{ m_{ij}\operatorname{Int}(f^{-1}(B)) - f^{-1}(\sigma_i \operatorname{Int}(B)) \} \\
&= \cup_{B \in \mathcal{P}(y)} \{ f^{-1}(\sigma_i \operatorname{Cl}(B)) - m_{ij}\operatorname{Cl}(f^{-1}(B)) \}.
\end{aligned}
$$

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Jour. Pure Math., Vol. 28, 2013, pp. 43-49

SAME FIXED POINT THEOREMS WITH G-ITERATION IN NORMED LINEAR SPACES

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ABSTRACT : In the present paper, a new type of one step iteration for self mapping is introduced and studied with a contractive type condition. The results obtained in this paper extend and improve the corresponding results of Dhage [2] and Sahu [7]. An illustrative example is given in support of our theorem.

Key words : Quasi contractive mapping, Normed linear space, Fixed point.
Mathematics Subject Classification. 47H10, 54H25.

1. PRELIMINARIES

Theorem 1.1 : Dhage [2] has proved a fixed point theorem satisfying the inequality

$$Tx - Ty \leq a(x - Tx + y - Ty)$$
$$+ \ (1-2a)\max\left\{x-y,\ x-Ty,\ y-Ty, \frac{1}{2}(x-Tx+y-Ty), \frac{1}{2}(x-Ty+y-Tx)\right\}$$

Definition 1.1 : Let X be a normed space and $T : X \to X$ is a self mapping then T is said to satisfy a Lipschitz condition with constant q if

$$Tx - Ty \leq q\,(x - y) \quad \forall x,\ y \in X$$

If $q < 1$ the T is called a contraction mapping.

Definition 1.2 : Let X be a normed space. Then a self mapping T of X is called quasi contractive mapping if

$$Tx - Ty \leq q \max\{x-y,\ x-Tx,\ y-Ty,\ x-Ty,\ y-Tx\} \ \forall x, y \in X, \ where \ 0 < q < 1.$$

Definition 1.3: Let X be a normed space. Then T_1 and T_2 two self mappings of X are called quasi contractive pair of mappings if

$$T_1x - T_2y \leq q \max\{x-y,\ x-T_1x,\ y-T_2y,\ x-T_2y,\ y-T_1x\} \ \forall x, y \in X, \ where \ 0 < q < 1.$$

2. MAIN RESULTS

Theorem 2.1. Let X be a closed convex subset of normed linear space N and let T be a quasi contractive self mapping of X and $\{x_n\}$ be the sequence of G-iterates associated with T. Then G-iteration process is defined in the following manner:

Let $x_0, x_1 \in X$ *and*

$$x_{n+2} = (\mu_n - \lambda_n)x_{n+1} + \lambda_n Tx_{n+1} + (1 - \mu_n - \lambda_n + k_n)Tx_n + (\lambda_n - k_n)x_n \textit{ for } n \geq 0$$

where $\{\mu_n\}$, $\{\lambda_n\}$ and $\{k_n\}$ satisfying

(i) $\mu_n = \lambda_n = k_n = 1$ *if* $n = 0$.

(ii) $0 < \lambda_n < 1,\ 0 < k_n < 1$ for $n > 0$.

(iii) $\mu_n \geq \lambda_n,\ \mu_n \geq k_n$ for $n \geq 0$.

(iv) $\lim_{n\to\infty} \lambda_n = h = \lim_{n\to\infty} k_n$ where $h > 0$.

(v) $\lim_{n\to\infty} \mu_n = 1$

If $\{x_n\}$ converges in X then it converges to a fixed point of T.

Proof : If $\{x_n\}$ converges on $z \in X$ then $\lim\limits_{n\to\infty} x_n = z$.

Now we shall show that z is the fixed point of T.

Consider

$$\begin{aligned}
z - Tz &\leq z - x_{n+2} + x_{n+2} - Tz \\
&\leq z - x_{n+2} + (\mu_n - \lambda_n)x_{n+1} + \lambda_n Tx_{n+1} + (1 - \mu_n - \lambda_n + k_n)Tx_n + (\lambda_n - k_n)x_n - Tz \\
&\leq z - x_{n+2} + (\mu_n - \lambda_n)x_{n+1} - Tz + \lambda_n\ Tx_{n+1} - Tz + (1 - \mu_n - \lambda_n + k_n)Tx_n \\
&\qquad - Tz + (\lambda_n - k_n)x_n - Tz \\
&\leq z - x_{n+2} + (\mu_n - \lambda_n)x_{n+1} - Tz \\
&\quad + \lambda_n q \max\{x_{n+1} - z,\ x_{n+1} - Tx_{n+1},\ z - Tz,\ x_{n+1} - Tz,\ z - Tx_{n+1}\} \\
&\quad + (1 - \mu_n - \lambda_n + k_n)Tx_n - Tz + (\lambda_n - k_n)x_n - Tz \qquad \text{...(2.1.1)}
\end{aligned}$$

We observe that

$$x_{n+1} - Tx_{n+1} = \frac{1}{\lambda_n} x_{n+1} - x_{n+2} + \frac{(1-\mu_n)}{\lambda_n} x_{n+1} - Tx_n + \frac{(\lambda_n - k_n)}{\lambda_n} x_n - Tx_n$$

And

$$z - Tx_{n+1} \le z - x_{n+1} + x_{n+1} - Tx_{n+1}$$

$$\le z - x_{n+1} + \frac{1}{\lambda_n} x_{n+1} - x_{n+2} + \frac{(1-\mu_n)}{\lambda_n} x_{n+1} - Tx_n + \frac{(\lambda_n - k_n)}{\lambda_n} x_n - Tx_n$$

Now putting above values in (3.1.1) then we have

$$z - Tz \le z - x_{n+2} + (\mu_n - \lambda_n)x_{n+1} - Tz$$

$$+ \lambda_n q \max\left\{\begin{array}{l} x_{n+1} - z, \frac{1}{\lambda_n} x_{n+1} - x_{n+2} + \frac{(1-\mu_n)}{\lambda_n} x_{n+1} - Tx_n + \frac{(\lambda_n - k_n)}{\lambda_n} x_n - Tx_n, z - Tz, \\ x_{n+1} - Tz, z - x_{n+1} + \frac{1}{\lambda_n} x_{n+1} - x_{n+2} + \frac{(1-\mu_n)}{\lambda_n} x_{n+1} - Tx_n + \frac{(\lambda_n - k_n)}{\lambda_n} x_n - Tx_n \end{array}\right\}$$

$$+ (1 - \mu_n - \lambda_n + k_n)Tx_n - Tz + (\lambda_n - k_n)x_n - Tz$$

Letting $n \to \infty$ then we have

$$z - Tz \le (1 - h + hq)z - Tz$$

which is a contradiction.

Hence $z = Tz$ is a fixed point of T.

Remark : When $\{\mu_n\} = \{1\}$ and $\{\lambda_n\} = \{k_n\}$ then above G-iterative process reduces to Mann iteration.

Theorem 2.2 Let X be a closed convex subset of normed linear space N and let T_1 *and* T_2 be quasi contractive pair of self mappings of X and $\{x_n\}$ be the sequence of G-iterates associated with T_1 & T_2; then G-iteration process is defined in the following manner:

Let $x_0, x_1 \in X$ and

$$x_{2n+2} = (\mu_n - \lambda_n)x_{2n+1} + \lambda_n T_1 x_{2n+1} + (1 - \mu_n - \lambda_n + k_n)T_2 x_{2n} + (\lambda_n - k_n)x_{2n}$$

and $$x_{2n+3} = (\mu_n - \lambda_n)x_{2n+2} + \lambda_n T_2 x_{2n+2} + (1 - \mu_n - \lambda_n + k_n)T_1 x_{2n+1} + (\lambda_n - k_n)x_{2n+1},$$

for $n \geq 0$

where $\{\mu_n\}$, $\{\lambda_n\}$ and $\{k_n\}$ satisfying

(i) $\mu_n = \lambda_n = k_n = 1$ if $n = 0$.

(ii) $0 < \lambda_n < 1,\ 0 < k_n < 1$ for $n > 0$.

(iii) $\mu_n \geq \lambda_n,\ \mu_n \geq k_n$ for $n \geq 0$.

(iv) $\lim_{n\to\infty} \lambda_n = h = \lim_{n\to\infty} k_n$ where $h > 0$.

(v) $\lim_{n\to\infty} \mu_n = 1$.

If $\{x_n\}$ converges to z in X then z is the common fixed point of T_1 and T_2.

Proof : If $\{x_n\}$ converges on $z \in X$ then $\lim\limits_{n\to\infty} x_n = z$.

Now we shall show that z is the fixed point of T.

Consider

$$\begin{aligned}
z - T_1 z &\leq z - x_{2x+3} + x_{2n+3} - T_1 z \\
&\leq z - x_{2n+3} + (\mu_n - \lambda_n)x_{2n+2} + \lambda_n T_2 x_{2n+2} + (1 - \mu_n - \lambda_n + k_n)T_1 x_{2n+1} \\
&\qquad + (\lambda_n - k_n)x_{2n+1} - T_1 z \\
&\leq z - x_{2n+3} + (\mu_n - \lambda_n)x_{2n+2} - T_1 z + \lambda_n T_2 x_{2n+2} - T_1 z + (1 - \mu_n - \lambda_n + k_n)T_1 x_{2n+1} \\
&\qquad - T_1 z + (\lambda_n - k_n)x_{2n+1} - T_1 z \\
&\leq z - x_{2n+3} + (\mu_n - \lambda_n)x_{2n+2} - T_1 z + \lambda_n q \max\{x_{2n+2} - z,\ x_{2n+2} - T_2 x_{2n+2}, \\
&\qquad z - T_1 z,\ z - T_2 x_{2n+2},\ x_{2n+2} - T_1 z\} + (1 - \mu_n - \lambda_n + k_n)T_1 x_{2n+1} \\
&\qquad - T_1 z + (\lambda_n - k_n)x_{2n+1} - T_1 z \qquad \text{...(2.2.1)}
\end{aligned}$$

We observe that

$$x_{2n+2} - T_2x_{2n+2} = \frac{1}{\lambda_n}x_{2n+2} - x_{2n+3} + \frac{(1-\mu_n)}{\lambda_n}x_{2n+2} - T_1x_{2n+1}\frac{(\lambda_n - k_n)}{\lambda_n}x_{2n+1} - T_1x_{2n+1}$$

and

$$z - T_2x_{2n+2} \leq z - x_{2n+2} + x_{2n+2} - T_2x_{2n+2}$$

$$\leq z - x_{2n+2} + \frac{1}{\lambda_n}x_{2n+2} - x_{2n+3} + \frac{(1-\mu_n)}{\lambda_n}x_{2n+2} - T_1x_{2n+1} + \frac{(\lambda_n - k_n)}{\lambda_n}x_{2n+1} - T_1x_{2n+1}$$

Now putting above values in (2.2.1) then we have

$$z - T_1z \leq z - x_{2n+3} + (\mu_n - \lambda_n)\, x_{2n+2} - T_1z$$

$$+\lambda_n q \max\left\{\begin{array}{l} x_{2n+2} - z, z - T_1z, \frac{1}{\lambda_n}x_{2n+2} - x_{2n+3} + \frac{(1-\mu_n)}{\lambda_n}x_{2n+2} - T_1x_{2n+1} + \frac{(\lambda_n - k_n)}{\lambda_n}x_{2n+1} - T_1x_{2n+1}, \\ z - x_{2n+2} + \frac{1}{\lambda_n}x_{2n+2} - x_{2n+3} + \frac{(1-\mu_n)}{\lambda_n}T_1x_{2n+1} - x_{2n+2} + \frac{(\lambda_n - k_n)}{\lambda_n}x_{2n+1} - T_1x_{2n+1}, \\ x_{2n+2} - T_1z \end{array}\right\}$$

$$+ (1 - \mu_n - \lambda_n + k_n)T_1x_{2n+1} - T_1z + (\lambda_n - k_n)x_{2n+1} - T_1z$$

Letting $n \to \infty$ then we have

$$z - T_1z \leq (1 - h + hq)z - T_1z$$

Which is a contradiction.

Hence $z = T_1z$ i.e. z is a fixed point of T_1.

Similarly we can show that

$$z - T_2z \leq (1 - h + hq)\, z - T_2z$$

Hence $z = T_2z$ i.e. z is a fixed point of T_2.

Finally we can say that z is a common fixed point of T_1 & T_2.

This completes the proof of theorem.

Example : Consider $N = R^3$, where R^3 is the set of all 3-tuples $x = (x_1, x_2, x_3)$ of real numbers and the norm x is defined by

$$x = \left\{\sum_{i=1}^{3} x_i^2\right\}^{\frac{1}{2}}, \quad \forall x \in R^3$$

let $X = \{x : x - 0 \leq 1,\ 0, x \in R^3\}$ and T be a self mapping of X into itself such that for any arbitrary $x = (x_1, x_2, x_3) \in X$ and $Tx = \left(\frac{x_1}{5}, \frac{x_2}{6}, \frac{x_3}{4}\right)$

Let $\{x_n\}$ be a sequence of clements of X such that

$x_{n+2} = (\mu_n - \lambda_n)x_{n+1} + \lambda_n Tx_{n+1} + (1 - \mu_n - \lambda_n + k_n)Tx_n + (\lambda_n - k_n)x_n$ *for* $n \geq 0$ and

$$\{\mu_n\} = \left\{\frac{2n+3}{2n+4}\right\} \quad \textit{for } n \geq 1 \textit{ and } \mu_0 = 1.$$

$$\{\lambda_n\} = \left\{\frac{n+1}{2n+1}\right\} \quad \textit{for } n \geq 0.$$

$$\{k_n\} = \left\{\frac{4n-3}{8n-3}\right\} \quad \textit{for } n \geq 0.$$

Consider $x_0 = (0, 0, 0)$, $x_1 = \left(\frac{1}{2}, 0, 0\right) \in X$; *then* $x_2 = \left(\frac{1}{10}, 0, 0\right)$, $x_3 = \left(\frac{4}{15}, 0, 0\right)$ *etc.*

Now taking $x = x_3 = \left(\frac{4}{15}, 0, 0\right)$ *and* $y = x_2 = \left(\frac{1}{10}, 0, 0\right)$ then we have

$$\frac{5}{37} \leq q < 1$$

Hence definition 1.2 is satisfied.

Clearly we can see that $Tx = x$ is satisfied only when $x = (0, 0, 0)$ i.e. 0 is the unique fixed point of T.

Remark : By using of Dhage's [2] condition in above example for $x = x_3$ *and* $y = x_2$ we have

$$\frac{16}{15} \leq a.$$

Which is a contradiction, since $a < 1$.

Hence condition of Dhage [2] fails. Therefore our results are genuine generalization of Dhage's [2] condition.

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Jour. Pure Math., Vol. 28, 2013, pp. 51-58

A SYSTEM OF LINEAR HOMOGENEOUS FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS SOLVED WITH THE HELP OF A NONLINEAR ORDINARY DIFFERENTIAL EQUATION

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ABSTRACT : In [1], three methods of finding the solutions of systems of linear homogeneous first-order ordinary differential equations (SODEs) with variable coefficients have been developed. The present article deals with a particular system of two linear homogeneous first-order ordinary differential equations, which is not amenable to the methods discussed there. An alternative approach to solving the given system has been developed here. The approach is based upon the solution of a corresponding Riccati Differential Equation, which is notably a nonlinear ordinary differential equation (ODE).

Key words : Systems of linear first-order ordinary differential equations with variable coefficients; Systems of linear algebraic equations, Exact differential equations, Riccati differential equations.
AMS Classification. 34B.

1. INTRODUCTION

A system of two linear homogeneous first-order ODEs comprises

$$\ell_{11}(t)x_1'(t)+\ell_{12}(t)x_2'(t)=a_{11}(t)x_1(t)+a_{12}(t)x_2(t), \qquad (t \in \mathrm{I}) \qquad (1.1a)$$

$$\ell_{21}(t)x_1'(t)+\ell_{22}(t)x_2'(t)=a_{21}(t)x_1(t)+a_{22}(t)x_2(t), \qquad (t \in \mathrm{I}) \qquad (1.1b)$$

where ℓ_{ij}, a_{ij} : I (interval) $\rightarrow$ C (set of complex numbers), $(i, j = 1, 2)$, are assumed to be continuous, $'$denotes differentiation with respect to t.

In approach I discussed in [1], the aim is to find suitable linear combinations of (1.1a), (1.1b), which can be converted to an exact differential equation in a new variable.

Writing

$$\left.\begin{array}{lll} \ell_{ij}(t) = \ell_{ij}, \;\; a_{ij}(t) = a_{ij}, & x_i(t) = x_i, & (i, j = 1, 2), \\ \underline{r} = \underline{r(t)} = (x_1(t), x_2(t))^T, & \underline{r}' = \underline{r'(t)} = \left(x_1'(t), x_2'(t)\right)^T, & \\ \underline{L}_i = (\ell_{i1}, \ell_{i2}), & \underline{A}_i = (a_{i1}, a_{i2})^T, & (t \in I), \end{array}\right\} \tag{1.2}$$

the SODEs (1.1a) – (1.1b) can be exhibited as

$$\underline{r}'^T . \underline{L}_i = \underline{r}^T . \underline{A}_i, \qquad (t \in I), \qquad (i = 1, 2). \tag{1.3}$$

Functions $\lambda_i : I \to C$ $(i = 1, 2)$ are sought for so that the ODE

$$\lambda_1\left(\underline{r'^T}.\underline{L}_1\right) + \lambda_2\left(\underline{r'^T}.\underline{L}_2\right) = \lambda_1\left(\underline{r^T}.\underline{A}_1\right) + \lambda_2\left(\underline{r^T}.\underline{A}_2\right) \tag{1.4}$$

can be converted to an exact ODE.

The ODE (1.4), rewritten in the form

$$\left[\underline{r}^T.(\lambda_1\underline{L}_1 + \lambda_2\underline{L}_2)\right]' = \underline{r}^T.\left[(\lambda_1\underline{L}_1 + \lambda_2\underline{L}_2)' + (\lambda_1\underline{A}_1 + \lambda_2\underline{A}_2)\right] \qquad (t \in I), \tag{1.5}$$

is convertible to a linear ODE if and only if there exists

$g : I \to C$ such that

$$(\lambda_1\underline{L}_1 + \lambda_2\underline{L}_2)' + (\lambda_1\underline{A}_1 + \lambda_2\underline{A}_2) = g(\lambda_1\underline{L}_1 + \lambda_2\underline{L}_2), \qquad (t \in I). \tag{1.6}$$

Writing

$$\mu_i = \mu_i(t) = \lambda_i(t)\exp\left(-\int_\alpha^t g(s)ds\right), \qquad (t \in I), \quad (i = 1, 2), \tag{1.7}$$

where $\alpha(\varepsilon\ I)$ is being fixed arbitrarily, (1.6) can be reduced to

$$\left(\mu_1\underline{L}_1 + \mu_2\underline{L}_2\right)' + \left(\mu_1\underline{A}_1 + \mu_2\underline{A}_2\right) = 0, \qquad (t \in I). \tag{1.8}$$

Now, (1.8) **is integrable if** there exists $h : I \to C$ such that

$$\mu_1\underline{A}_1 + \mu_2\underline{A}_2 = h\left[\mu_1\underline{L}_1 + \mu_2\underline{L}_2\right], \qquad (t \in I). \tag{1.9}$$

Notably the same function h is required for both the scalar ODEs of the vector ODE (1.9) so that the scalar ODE (1.5) can be solved.

The equations (1.9), being a system of two linear homogeneous algebraic equations in the unknowns, μ_1, μ_2, will determine $(\mu_1, \mu_2) \neq (0, 0)$ if

$$\begin{vmatrix} a_{11} - h\ell_{11} & a_{21} - h\ell_{21} \\ a_{12} - h\ell_{12} & a_{22} - h\ell_{22} \end{vmatrix} = 0, \qquad (t \in I), \tag{1.10}$$

(1.10), being a quadratic equation in h, determines, in general, two functions h_1, h_2. Substituting $h = h_i$ $(i = 1, 2)$ in (1.9), its corresponding solutions for μ_1, μ_2 are denoted by μ_{i1}, μ_{i2} $(i = 1, 2)$.

If this (μ_{i1}, μ_{i2}) *satisfies the SODEs* (1.8) for (μ_1, μ_2), one obtains, using (1.9),

$$\left(\mu_{i1}\underline{L}_1 + \mu_{i2}\underline{L}_2\right)' + h_i\left(\mu_{i1}\underline{L}_1 + \mu_{i2}\underline{L}_2\right) = 0, \qquad (t \in I), \quad (i = 1, 2), \tag{1.11}$$

whence it can be deduced that

$$\mu_{i1}\underline{L}_1 + \mu_{i2}\underline{L}_2 = \left(C_{i1}, C_{i2}\right)^T \exp\left(-\int_a^t h_i(s)\,ds\right), \qquad (t \in I), \quad (i = 1, 2), \tag{1.12}$$

where C_{i1}, C_{i2} $(\in \mathbb{C})$ are the parameters of integration and $a(\in I)$ is chosen arbitrarily. Noting that $\lambda_1 : \lambda_2 = \mu_1 : \mu_2$ [vide (1.7)] and (1.4) is homogeneous in λ_1, λ_2, substituting μ_{i1}, μ_{i2} for λ_1, λ_2 in (1.4) and using (1.12) one gets

$$\underline{r}'^T.(C_{i1}, C_{i2})^T = \underline{r}^T.h_i(C_{i1}, C_{i2})^T, \qquad (t \in I), \quad (i = 1, 2). \tag{1.13}$$

Hence

$$C_{i1}x_1 + C_{i2}x_2 = D_i \exp\left(\int_a^t h_i(s)\,ds\right), \qquad (t \in I), \quad (i = 1, 2), \tag{1.14}$$

where $D_i \in \mathbb{C}$ $(i = 1, 2)$ are the parameters of integration.

The required solutions of the SODEs (1.1a) – (1.1b) are then obtained by solving the algebraic equations (1.14).

The trouble starts when the μ_{i1}, μ_{i2}, obtained from (1.9) for a root h_i of (1.10), fails to satisfy the SODEs (1.8). In this case, the Approach I of [1] fails.

In the sequel, §2 presents a method of obtaining $\lambda_1 : \lambda_2 = \mu_1 : \mu_2$ so that (1.8) is satisfied.

In §3, a particular SODEs has been cited for which none of the Approaches, I, II, III of [1] works.

In §4, the method presented in §2 is applied to the SODEs under purview to obtain its solutions.

2. THE METHOD

The two scalar equations in (1.6) are

$$(\lambda_1\ell_{11} + \lambda_2\ell_{21})' + (\lambda_1 a_{11} + \lambda_2 a_{21}) = g(\lambda_1\ell_{11} + \lambda_2\ell_{21}),$$

$$(\lambda_1\ell_{12} + \lambda_2\ell_{22})' + (\lambda_1 a_{12} + \lambda_2 a_{22}) = g(\lambda_1\ell_{12} + \lambda_2\ell_{22}),$$

whence one gets

$$\frac{(\lambda_1\ell_{11} + \lambda_2\ell_{21})' + (\lambda_1 a_{11} + \lambda_2 a_{21})}{(\lambda_1\ell_{12} + \lambda_2\ell_{22})' + (\lambda_1 a_{12} + \lambda_2 a_{22})} = \frac{\lambda_1\ell_{11} + \lambda_2\ell_{21}}{\lambda_1\ell_{12} + \lambda_2\ell_{22}}, \quad (t \in I). \tag{1.15}$$

The aim is to find $\lambda_1 : \lambda_2$ from (1.15).

As any linear SODEs (1.1a) – (1.1b) can be converted to an SODEs with $\ell_{12} = \ell_{21} = 0$, without loss of generality it may be assumed that $\ell_{12} = \ell_{21} = 0$ in (1.1a) – (1.1b).

Then (1.15) becomes

$$(\lambda_1\ell_{11})'(\lambda_2\ell_{22}) - (\lambda_1\ell_{11})(\lambda_2\ell_{22})' = \lambda_1\ell_{11}(\lambda_1 a_{12} + \lambda_2 a_{22}) - \lambda_2\ell_{22}(\lambda_1 a_{11} + \lambda_2 a_{21})$$

$$\text{or,}\quad \left(\frac{\lambda_1\ell_{11}}{\lambda_2\ell_{22}}\right)' = \left(\frac{\lambda_1}{\lambda_2}\right)^2 \frac{\ell_{11}a_{12}}{\ell_{22}^2} + \frac{\lambda_1}{\lambda_2}\left(\frac{\ell_{11}a_{22}}{\ell_{22}^2} - \frac{a_{11}}{\ell_{22}}\right) - \frac{a_{21}}{\ell_{22}}, \quad (t \in I). \tag{1.16}$$

The required $\lambda_1 : \lambda_2$ is then obtained from the solutions of the ODE (1.16).

Note : Attention may be drawn to the fact that (1.16) is a nonlinear ODE, known as Riccati differential equation. Thus the above derivation cites an instance, where a nonlinear ODE provides the requisite clue to solving some linear homogeneous SODEs, although the common practice is to use solutions of linear ODEs as approximations to solutions of nonlinear ODEs.

3. A PARTICULAR SODEs

$$t\left(3t^2-1\right)x_1' = \left(6t^2-1\right)x_1 + t^2x_2, \quad (t \in I), \tag{ia}$$

$$t\left(3t^2-1\right)x_2' = 3x_1 + 3t^2x_2, \quad (t \in I), \tag{ib}$$

where $x_i : I =]0, \frac{1}{\sqrt{3}}[\to \mathbf{R}$ (set of real numbers), $i = 1, 2$.

Clearly the Approaches II, III of [1] are not applicable to the system $(ia) - (ib)$.

In this case the determinantal equation (1.10) becomes

$$\begin{vmatrix} \left(6t^2-1\right) - ht\left(3t^2-1\right) & 3 \\ t^2 & 3t^2 - ht\left(3t^2-1\right) \end{vmatrix} = 0, \qquad (t \in I)$$

or,

$$6t^2 - ht(9t^2-1) + h^2t^2(3t^2-1) = (1-ht)\{6t^2 - ht(3t^2-1)\} = 0, \quad (t \in I). \tag{ii}$$

Clearly the two roots of (ii) are $\quad h_1 = \frac{1}{t}, \quad h_2 = \frac{6t}{3t^2-1}.$ (iii)

It is shown below that the root h_2 leads to the ratio $\lambda_1 : \lambda_2 = \lambda_{21} : \lambda_{22}$ so that $\lambda_{21}(ia) + \lambda_{22}(ib)$ becomes integrable, but the ratio $\lambda_1 : \lambda_2 = \lambda_{11} : \lambda_{12}$ obtained from the root h_1 fails to do so.

Let $h = h_2 = \frac{6t}{3t^2-1}$ in (1.9).

Then both the equations of (1.9) becomes $-\mu_1 + 3\mu_2 = 0.$ (iv)

So $\lambda_1 : \lambda_2 = \mu_1 : \mu_2 = 3 : 1$.

Adding 3 times (ia) to (ib) one obtains

$$t\left(3t^2 - 1\right)\left(3x_1' + x_2'\right) = 6t^2\left(3x_1 + x_2\right). \tag{v}$$

From (v), on integration, one obtains

$$3x_1 + x_2 = C_1\ (3t^2 - 1), \quad (t \in I), \tag{vi}$$

where C_1 ($\in \mathbb{R}$) is the parameter of integration.

Using (vi) in any one of the equations (ia), (ib), one obtains an ODE in a single variable. As for example, substituting for x_1 from (vi) in (ib) one gets

$$tx_2' = C_1 + x_2,$$

whence, on integration, one derives $x_2 = -C_1 + C_2 t, \quad (t \in I),$ (vii)

where C_2 ($\in \mathbb{R}$) is the parameter of integration.

From (vi) and (vii) one obtains $x_1 = C_1 t^2 - \frac{1}{3} C_2 t, \quad (t \in I).$ (viii)

Thus the SODEs (ia) – (ib) is solved.

However, taking $h = h_1 = \frac{1}{t}$ in (1.9), both of its equations become $t^2\mu_1 + \mu_2 = 0$.

So $\lambda_1 : \lambda_2 = \mu_1 : \mu_2 = 1 : -t^2$.

Now, subtracting t^2 times (ib) from (ia), one obtains

$$tx_1' - x_1 = t^2\left(tx_2' - x_2\right), \quad (t \in I),$$

which, unfortunately, is not convertible to an exact ODE.

Note : The failure of getting a proper set for $\lambda_1 : \lambda_2 = \mu_1 : \mu_2$, using $h = \frac{1}{t}$, so that $\lambda_1(ia) + \lambda_2(ib)$ becomes integrable, can be traced back to the failure of the solution $(\mu_1, \mu_2) = (1, -t^2)$ of (1.9) with $h = \frac{1}{t}$ in satisfying the SODEs (1.8) corresponding to the problem under concern.

4. APPLICATION OF THE METHOD OF §2 TO THE SODEs (ia) – (ib) :

For the SODEs (*ia*) – (*ib*), the equation (1.16) becomes

$$\left(\frac{\lambda_1}{\lambda_2}\right)' = \left(\frac{\lambda_1}{\lambda_2}\right)^2 \frac{t}{3t^2-1} - \frac{1}{t}\frac{\lambda_1}{\lambda_2} - \frac{3}{t\left(3t^2-1\right)}$$

or, $$\left(\frac{t\lambda_1}{\lambda_2}\right)' = \left(\frac{t\lambda_1}{\lambda_2}\right)^2 \frac{1}{3t^2-1} - \frac{3}{3t^2-1}, \qquad (t \in I)$$

or, $u' = \dfrac{u^2-3}{3t^2-1}$, where, $u = \dfrac{t\lambda_1}{\lambda_2}$. $(t \in I)$.

Hence, on integration, one obtains

$$\frac{u-\sqrt{3}}{u+\sqrt{3}} = C\frac{t\sqrt{3}-1}{t\sqrt{3}+1}, \qquad (t \in I), \tag{ix}$$

where $C(\in \mathbf{R})$ is the parameter of integration.

With $C = 1$ one gets $u = 3t$, so that $\lambda_1 : \lambda_2 = 3 : 1$. (x)

This ratio $3 : 1$ for $\lambda_1 : \lambda_2$ has been obtained on taking

$$h = h_2 = \frac{6t}{3t^2-1}.$$

With $C = -1$ one gets $u = \dfrac{1}{t}$, so that $\lambda_1 : \lambda_2 = 1 : t^2$.

It is observed that, adding t^2 times (ib) to (ia) one obtains

$$t\left(3t^2-1\right)\left(x_1' + t^2x_2'\right) = \left\{\left(6t^2-1\right) + 3t^2\right\}x_1 + \left(t^2 + 3t^4\right)x_2,$$

which can be rewritten as

$$\left(x_1 + t^2x_2\right)' = \frac{9t^2-1}{t\left(3t^2-1\right)}\left(x_1 + t^2x_2\right), \qquad (t \in I). \tag{xi}$$

Hence, on integration, one derives

$$x_1 + t^2x_2 = C_2 t\,(3t^2-1), \qquad (t \in I), \tag{xii}$$

where C_2 ($\in \mathbb{R}$) is the parameter of integration.

Solving (vi) and (xii), the required solutions of (ia) – (ib) are obtained as

$$x_1 = C_1 t^2 - C_2 t, \qquad x_2 = -C_1 + 3C_2 t, \qquad (t \in I),$$

C_1, C_2 ($\in \mathbb{R}$) being the parameters.

5. REMARKS

(a) The Method presented here is applicable to SODEs with *two* dependent variables only.

(b) Any solution of the Riccati differential equation (1.16) provides a ratio $\lambda_1 : \lambda_2$ which can be used to determine a linear combination of the equations (1.1a), (1.1b), that leads to an exact differential equation. Hence there exist infinite number of linear combinations of the equations (1.1a), (1.1b), each leading to an exact differential equation. All such linear combinations are determined by the solutions of the Riccati differential equation (1.16).

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Jour. Pure Math., Vol. 28, 2013, pp. 59-72

ON SOLUTIONS OF A LINEAR SECOND-ORDER HOMOGENEOUS ORDINARY DIFFERENTIAL EQUATION DETERMINED BY A GIVEN QUADRATIC HOMOGENEOUS BOUNDARY CONDITION

JAYASRI SETT & SUSMITA DAS

ABSTRACT : In this paper we determine those members of the solution space of the second-order linear Homogeneous Ordinary Differential Equation (ODE)

$$p_0(x)y''(x) + p_1(x)y''(x) + p_2(x)y(x) = 0, \quad x \in [a, b],$$

where p_0, p_1, p_2 are complex – valued continuous functions on $[a, b]$ and $p_0(x) \neq 0$ for any $x \in [a, b]$, which satisfy a given Quadratic Homogeneous Boundary Condition (QBC) of the form

$$U_\alpha[y] = \alpha_1 y^2(a) + \alpha_2 y^2(a) + \alpha_3 y^2(b) + \alpha_4 y'^2(b) = 0,$$

where α_1, α_2, α_3 and α_4 are real numbers.

Let ξ and η be the solutions of the ODE which satisfy the initial conditions

$$\xi(a) = 1, \qquad \xi'(a) = 0,$$

$$\eta(a) = 0, \qquad \eta'(a) = 1,$$

and let

$$\xi(b) = \xi_1 + i\xi_2, \qquad \xi'(b) = \xi'_1 + i\xi'_2,$$

$$\eta(b) = \eta_1 + i\eta_2, \qquad \eta'(b) = \eta'_1 + i\eta'_2,$$

where ξ_1, ξ_2, ξ'_1, ξ'_2, η_1, η_2, η'_1, η'_2 are real numbers.

Let S_L be the linear space spanned by ξ and η. Clearly $S_L = \{\alpha\xi + \beta\eta, \alpha,\beta \in C\}$. For each $\ell \in C(\ell = u + iv)$ we consider all those members of S_L which are of the form $\lambda(\xi + \ell\eta)$, $\lambda \in C$. Let $\xi + \ell\eta$ be chosen as the representative member of the class $\{\lambda(\xi + \ell\eta) : \lambda \in C\}$ and η be the representative member of the class $\{\lambda\eta : \lambda \in C\}$. With this understanding S_L can be represented as $\mathbf{s}_L = \{\eta, \xi + \ell\eta, \ell \in C\}$.

We show that,

if **(I)** $\mathbf{U_\alpha[\eta] \neq 0}$, then there exist two members of $\mathbf{s}_L$ of the form $\boldsymbol{\xi + (u_1 + iv_1)\eta}$ and $\boldsymbol{\xi + (u_2 + iv_2)\eta}$ which satisfy the given QBC. (u_1, v_1) and (u_2, v_2) are two distinct points of intersection of two rectangular orthogonal hyperbolas having the same centre O and are symmetric w.r.to O;

If (ii) $U_\alpha[\eta] = 0$, $U_\alpha[\xi] \neq 0$ then there exist two members of s_L of the form η and $\xi + (u + iv)\eta$, where u and v are uniquely determined, which satisfy the given QBC;

if (iii) $U_\alpha[\eta] = 0 = U_\alpha[\xi]$, then two cases arise :

a) if the co-efficient determinant of

$$2u(\alpha_3\xi_1\eta_1 - \alpha_3\xi_2\eta_2 + \alpha_4\xi'_1\eta'_1 - \alpha_4\xi'_2\eta'_2) - 2v(\alpha_3\xi_2\eta_1 + \alpha_3\xi_1\eta_2 + \alpha_4\xi'_2\eta'_1 + \alpha_4\xi'_1\eta'_2) = 0$$

and

$$2u(\alpha_3\xi_2\eta_1 + \alpha_3\xi_1\eta_2 + \alpha_4\xi'_2\eta'_1 + \alpha_4\xi'_1\eta'_2) - 2v(-\alpha_3\xi_1\eta_1 + \alpha_3\xi_2\eta_2 - \alpha_4\xi'_1\eta'_1 + \alpha_4\xi'_2\eta'_2) = 0$$

is not zero then the only members of s_L satisfying the given QBC are ξ and η;

b) if the co-efficient determinant of the equations of (a) is zero then **all members of s_L** satisfy the given QBC.

1. INTRODUCTION

We consider the second-order linear homogeneous Ordinary Differential Equation (ODE)

$$p_0(x)y''(x) + p_1(x)y'(x) + p_2(x)y(x) = 0, \quad x \in [a, b], \tag{1.1}$$

where p_0, p_1, p_2 are complex – valued continuous functions on $[a, b]$ and $p_0(x) \neq 0$ for any $x \in [a, b]$.

Let ξ and η be the solutions of (1.1) which satisfy the initial conditions

$$\xi(a) = 1, \qquad \xi'(a) = 0, \tag{1.2}$$

$$\eta(a) = 0, \qquad \eta'(a) = 1, \tag{1.3}$$

The linear space spanned by ξ, η is the solution space S_L of all solutions of (1.1).

Clearly $S_L = \{\alpha\xi + \beta\eta,\ \alpha, \beta \in C\}$.

For each $\ell \in C$ $(\ell = u + iv)$ we consider all those members of S_L which are of the form $\lambda(\xi + \ell\eta)$, $\lambda \in C$. Let $\xi + \ell\eta$ be chosen as the representative member of the class $\{\lambda(\xi + \ell\eta) : \lambda \in C\}$ and η be the representative member of the class $\{\lambda\eta : \lambda \in C\}$. With this understanding S_L can be represented as

$$\mathbf{s}_L = \{\mathbf{v}, \xi + \ell\eta, \ell \in \mathbf{C}\}. \tag{1.4}$$

A Quadratic Homogeneous Boundary Condition (QBC) is of the form

$$U_\alpha[y] = \alpha_1 y^2(a) + \alpha_2 y'^2(a) + \alpha_3 y^2(b) + \alpha_4 y'^2(b) = 0, \tag{1.5}$$

where α_1, α_2, α_3 and α_4 are real numbers.

Let us write

$$\xi(b) = \xi_1 + i\xi_2, \qquad \xi'(b) = \xi'_1 + i\xi'_2, \tag{1.6}$$

$$\eta(b) = \eta_1 + i\eta_2, \qquad \eta'(b) = \eta'_1 + i\xi'_2, \tag{1.7}$$

where ξ_i, ξ'_i, η_i, η'_i, $(i = 1, 2)$ are real numbers.

In this paper we determine the members of $\mathbf{s}_L$ which satisfy a given boundary condition of the type (1.5). There are three cases and in cases (i) and (ii) it is found that there are two members of $\mathbf{s}_L$ which satisfy (1.5). In case (iii) all members of $\mathbf{s}_L$ satisfy (1.5). In §2 the theorem is stated and proved.

Examples are given in §3.

2. THEOREM

Let consider the ODE (1.1) with ξ and η as solutions which satisfy the conditions (1.2), (1.3), (1.6) and (1.7). Let $\mathbf{s}_L$ be defined as in (1.4). Let the QBC

$$U_\alpha[y] = \alpha_1 y^2(a) + \alpha_2 y'^2(a) + \alpha_3 y^2(b) + \alpha_4 y'^2(b) = 0 \text{ be given.}$$

If **(i)** $U_\alpha[\eta] \neq 0$, then there exist two members of $\mathbf{s}_L$ of the form $\xi + (u_1 + iv_1)\eta$ and $\xi + (u_2 + iv_2)\eta$ which satisfy the given QBC. (u_1, v_1) and (u_2, v_2) are two distinct points of intersection of two rectangular orthogonal hyperbolas having the same centre O and are symmetric w.r. to O;

if **(ii)** $U_\alpha[\eta] = 0$, $U_\alpha[\xi] \neq 0$, then there exist two members of $\mathbf{s}_L$ of the form η and $\xi + (u + iv)\eta$, where u and v are uniquely determined, which satisfy the given QBC;

if (iii) $U_\alpha[\eta] = 0 = U_\alpha[\xi]$, then two cases arise :

a) If the co-efficient determinant of

$$2u(\alpha_3\xi_1\eta_1 - \alpha_3\xi_2\eta_2 + \alpha_4\xi'_1\eta'_1 - \alpha_4\xi'_2\eta'_2) - 2v(\alpha_3\xi_2\eta_1 + \alpha_3\xi_1\eta_2 + \alpha_4\xi'_2\eta'_1 + \alpha_4\xi'_1\eta'_2) = 0$$

and

$$2u(\alpha_3\xi_2\eta_1 + \alpha_3\xi_1\eta_2 + \alpha_4\xi'_2\eta'_1 + \alpha_4\xi'_1\eta'_2) - 2v(-\alpha_3\xi_1\eta_1 + \alpha_3\xi_2\eta_2 - \alpha_4\xi'_1\eta'_1 + \alpha_4\xi'_2\eta'_2) = 0$$

is not zero, then the only members of $\mathbf{s}_L$ satisfying the given QBC are ξ and η;

b) if the co-efficient determinant of the equations of (a) is zero, then **all members for $\mathbf{s}_L$** satisfy the given QBC.

Proof : Let $\xi + \ell\eta$ satisfy (1.5), where $\ell = u + iv$. Substituting $\xi + \ell\eta$ into (1.5), and separating real and imaginary parts, we obtain

$$\begin{aligned} &u^2\left(\alpha_2 + \alpha_3\eta_1^2 - \alpha_3\eta_2^2 + \alpha_4\eta_1'^2 - \alpha_4\eta_2'^2\right) - v^2\left(\alpha_2 + \alpha_3\eta_1^2 - \alpha_3\eta_2^2 + \alpha_4\eta_1'^2 - \alpha_4\eta_2'^2\right) \\ &- 4uv(\alpha_3\eta_1\eta_2 + \alpha_4\eta'_1\eta'_2) + 2u(\alpha_3\xi_1\eta_1 - \alpha_3\xi_2\eta_2 + \alpha_4\xi'_1\eta'_1 - \alpha_4\xi'_2\eta'_2) \\ &- 2v(\alpha_3\xi_2\eta_1 + \alpha_3\xi_1\eta_2 + \alpha_4\xi'_2\eta'_1 + \alpha_4\xi'_1\eta'_2) \\ &\qquad + \left(\alpha_1 + \alpha_3\xi_1^2 - \alpha_3\xi_2^2 + \alpha_4\xi_1'^2 - \alpha_4\xi_2'^2\right) = 0 \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &u^2(2\alpha_3\eta_1\eta_2 + 2\alpha_4\eta'_1\eta'_2) - v^2(2\alpha_3\eta_1\eta_2 + 2\alpha_4\eta'_1\eta'_2) \\ &\quad -2uv\left(-\alpha_2 - \alpha_3\eta_1^2 + \alpha_3\eta_2^2 - \alpha_4\eta_1'^2 + \alpha_4\eta_2'^2\right) \\ &\quad + 2u(\alpha_3\xi_2\eta_1 + \alpha_3\xi_1\eta_2 + \alpha_4\xi'_2\eta'_1 + \alpha_4\xi'_1\eta'_2) - 2v(-\alpha_3\xi_1\eta_1 + \alpha_3\xi_2\eta_2 \\ &\quad - \alpha_4\xi'_1\eta'_1 + \alpha_4\xi'_2\eta'_2) + (2\alpha_3\xi_1\xi_2 + 2\alpha_4\xi'_1\xi'_2) = 0. \end{aligned} \tag{2.2}$$

The points of intersection (u, v) of (2.1) and (2.2) determine the members of s_L that will satisfy (1.5).

Comparing (2.1) with the equation of the conic

$$au^2 + bv^2 + 2huv + 2gu + 2fv + c = 0, \tag{2.3}$$

we see that

$$\left.\begin{aligned} a &= \alpha_2 + \alpha_3\eta_1^2 - \alpha_3\eta_2^2 + \alpha_4\eta_1'^2 - \alpha_4\eta_2'^2 = -b, \\ h &= -2(\alpha_3\eta_1\eta_2 + \alpha_4\eta'_1\eta'_2), \\ g &= (\alpha_3\xi_1\eta_1 - \alpha_3\xi_2\eta_2 + \alpha_4\xi'_1\eta'_1 - \alpha_4\xi'_2\eta'_2), \\ f &= -(\alpha_3\xi_2\eta_1 + \alpha_3\xi_1\eta_2 + \alpha_4\xi'_2\eta'_1 + \alpha_4\xi'_1\eta'_2), \\ c &= \alpha_1 + \alpha_3\xi_1^2 - \alpha_3\xi_2^2 + \alpha_4\xi_1'^2 + \alpha_4\xi_2'^2. \end{aligned}\right\} \tag{2.4}$$

Similarly comparing with the equation of the conic (2.3), (2.2) can be written in the form

$$-hu^2 + hv^2 + 2auv - 2uf + 2vg + c' = 0, \tag{2.5}$$

$$\text{where } c' = 2(\alpha_3\xi_1\xi_2 + \alpha_4\xi'_1\xi'_2). \tag{2.6}$$

(i) $\mathbf{U_\alpha[\eta] \neq 0}$

Separating real and imaginary parts, we get,

$$\left.\begin{aligned} &\alpha_2 + \alpha_3\eta_1^2 - \alpha_3\eta_2^2 + \alpha_4\eta_1'^2 - \alpha_4\eta_2'^2 \neq 0, \\ \text{or,}\quad &\alpha_3\eta_1\eta_2 + \alpha_4\eta'_1\eta'_2\ 0, \end{aligned}\right\} \tag{2.7}$$

or both not equal to zero.

Thus we see that if (2.7) holds, then either $a = -b \neq 0$ or $h \neq 0$ or both not equal to zero.

Hence (2.1) represents a rectangular hyperbola. For these results see [4], [7].

Using (2.4), the centre (A, B) of (2.1) is given by $A = \frac{hf - bg}{ab - h^2}$, $B = \frac{gh - af}{ab - h^2}$, where $\Delta = h^2 - ab = h^2 + a^2 > 0$.

Here

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= -c(h^2 + a^2) + 2fgh - a(f^2 - g^2).$$

Similarly (2.2) represents a rectangular hyperbola with centre (A_1, B_1) where using (2.4)

$$A_1 = \frac{ag + hf}{ab - h^2} = \frac{hf - bg}{ab - h^2} = A,$$

$$B_1 = \frac{gh - af}{ab - h^2} = B.$$

Using the condition of orthogonality we can show that (2.1) and (2.2) are orthogonal hyperbolas.

Transferring the origin from (0, 0) to (A, B), using the transformation

$u = u' + A, \quad v = v' + B$, (2.1) becomes

$$au'^2 + bv'^2 + 2hu'v' - \frac{D}{\Delta} = 0, \tag{2.8}$$

see [4], [7].

If now, the conic (2.8) is referred to the lines through (A, B) which are inclined at an angle γ to the original axes i.e., if now the transformation

$$\left.\begin{aligned} u' &= U\cos\gamma - V\sin\gamma \\ v' &= U\sin\gamma - V\cos\gamma \end{aligned}\right\} \tag{2.9}$$

is employed, equation (2.8) becomes ([4], [7])

$$a'U^2 + b'V^2 + 2h'UV - \frac{D}{\Delta} = 0, \tag{2.10}$$

where

$$\left.\begin{aligned}
a' &= a\cos^2\gamma + b\sin^2\gamma + 2h\cos\gamma\sin\gamma \\
&= a\cos^2\gamma - a\sin^2\gamma + 2h\cos\gamma\sin\gamma, \\
b' &= a\sin^2\gamma + b\cos^2\gamma - 2h\cos\gamma\sin\gamma \\
&= a\sin^2\gamma - a\cos^2\gamma - 2h\cos\gamma\sin\gamma \\
&= -(a\cos^2\gamma - a\sin^2\gamma + 2h\cos\gamma\sin\gamma) \\
&= -a',
\end{aligned}\right\} \tag{2.11}$$

$$\begin{aligned}
h' &= -2a\cos\gamma\sin\gamma + 2b\cos\gamma\sin\gamma - 2\mathrm{h}\sin^2\gamma + 2h\cos^2\gamma \\
&= -4\mathrm{a}\cos\gamma\sin\gamma - 2\mathrm{h}\sin^2\gamma + 2h\cos^2\gamma.
\end{aligned}$$

If angle γ is so chosen that

$$h' = 0, \tag{2.12}$$

equation (2.10) reduces to

$$a'U^2 + b'V^2 - \frac{D}{\Delta} = 0. \tag{2.13}$$

Using the same transformation as in conic (2.1), (2.2) now becomes

$$A'U^2 + B'V^2 + 2H'UV - \frac{D'}{\Delta} = 0, \tag{2.14}$$

where

$$\left.\begin{aligned}
A' &= -h\cos^2\gamma + h\sin^2\gamma + 2a\cos\gamma\sin\gamma, \\
B' &= -h\sin^2\gamma + h\cos^2\gamma - 2a\cos\gamma\sin\gamma \\
&= -A', \\
H' &= 4h\cos\gamma\sin\gamma - 2a\sin^2\gamma + 2a\cos^2\gamma \\
&= 2(a\cos^2\gamma - a\sin^2\gamma + 2h\cos\gamma\sin\gamma) \\
&= 2a'
\end{aligned}\right\} \tag{2.15}$$

$$= -2b' \text{ from (2.11).}$$

$$D' = \begin{vmatrix} -h & a & -f \\ a & h & g \\ -f & g & c' \end{vmatrix}$$

$$= -c'(h^2 + a^2) - 2fga + h - (f^2 - g^2)$$

Using (2.12) and (2.14), (2.15) becomes

$$2H'UV - \frac{D'}{\Delta} = 0$$

or, $$4a'UV - \frac{D'}{\Delta} = 0 \tag{2.16}$$

To find the points of intersection of (2.13) and (2.16), we proceed as follows:

we write $U = \dfrac{M}{V}$, (2.17)

where $M = \dfrac{D'}{4a\Delta}$ (from (2.16)) (2.18)

We consider the following two cases (in all cases $\Delta > 0$)

Case I : $\Delta > 0$, $D > 0$ and

Case II : $\Delta > 0$, $D < 0$.

Case I and II have the following subcases :

(i) $a' > 0$, $D' > 0$,

(ii) $a' > 0$, $D' < 0$,

(iii) $a' < 0$, $D' > 0$,

(iv) $a' < 0$, $D' < 0$.

Of these cases we discuss case (i) in detail :

(i) $a' > 0$, $D' > 0$

Then from (2.18) $M > 0$.

Substituting (2.17) in (2.13) we have

$$a'\left(\frac{M}{V}\right)^2 + b'V^2 - \frac{D}{\Delta} = 0$$

or, $b'\Delta V^4 - DV^2 + a'\Delta M^2 = 0.$

Let $f(V) = b'\Delta V^4 - DV^2 + a'\Delta M^2.$

It can be verified by change of sign of co-efficients that $f(V) = 0$ has one positive and one negative real root. In this case from (2.17), the corresponding Us are positive and negative respectively. Hence the real points of intersection of (2.13) and (2.16) are (U, V), and $(-U, -V)$ $((U > 0, V > 0)$ which are symmetric about O. The other two roots of $f(V) = 0$ are imaginary and the corresponding Us are also imaginary.

Similarly in all other cases we get, two real points of intersection (U_1, V_1) and (U_2, V_2) which are symmetric w.r.to O.

Reverting the origin (0, 0) to (A, B) we get two symmetric points of intersection (u_1, v_1) and (u_2, v_2) of (2.1) and (2.2) about (A, B). Hence two members of s_L of the form $\xi + (u_1 + iv_1)\eta$ and $\xi + (u_2 + iv_2)\eta$ satisfy (1.5).

(ii) $U_\alpha[\eta] = 0$, $U_\alpha[\xi] \neq 0$: then separating the real and imaginary parts of $U_\alpha[\xi]$ and $U_\alpha[\eta]$ respectively we get

$$\alpha_1 + \alpha_3\xi_1^2 - \alpha_3\xi_2^2 + \alpha_4\xi_1'^2 + \alpha_4\xi_2'^2 \neq 0 \text{ or } \alpha_3\xi_1\xi_2 + \alpha_4\xi'_1\xi'_2 \neq 0 \tag{2.19}$$

or both not equal to zero,

and

$$\alpha_2 + \alpha_3\eta_1^2 - \alpha_3\eta_2^2 + \alpha_4\eta_1'^2 - \alpha_4\eta_2'^2 = 0, \quad \alpha_3\eta_1\eta_2 + \alpha_4\eta'_1\eta'_2 = 0 \tag{2.20}$$

From (2.4) and (2.5)

$a = b = h = 0$, $c \neq 0$, or $c' \neq 0$, or both not equal to zero.

Then

$D = -c(h^2 + a^2) + 2fgh - a(f^2 - g^2) = 0,$

$D' = -c'(h^2 + a^2) - 2fgh + h(-f^2 + g^2) = 0.$

In this case (2.1) and (2.2) represent a pair of mutually perpendicular straight lines not passing through (0, 0).

From (2.1)

$$2u(\alpha_3\xi_1\eta_1 - \alpha_3\xi_2\eta_2 + \alpha_4\xi'_1\eta'_1 - \alpha_4\xi'_2\eta'_2)$$
$$- 2v(\alpha_3\xi_2\eta_1 + \alpha_3\xi_1\eta_2 + \alpha_4\xi'_2\eta'_1 + \alpha_4\xi'_1\eta'_2) + (\alpha_1 + \alpha_3\xi_1^2 - \alpha_3\xi_2^2$$
$$+ \alpha_4\xi'^2_1 - \alpha_4\xi'^2_2 = 0, \tag{2.21}$$

and from (2.2)

$$2u(\alpha_3\xi_2\eta_1 + \alpha_3\xi_1\eta_2 + \alpha_4\xi'_2\eta'_1 + \alpha_4\xi'_1\eta'_2) - 2v(- \alpha_3\xi_1\eta_1 + \alpha_3\xi_2\eta_2$$
$$- \alpha_4\xi'_1\eta'_1 + \alpha_4\xi'_2\eta'_2) + (2\alpha_3\xi_1\xi_2 + 2\alpha_4\xi'_1\xi'_2) = 0 \tag{2.22}$$

From (2.21) and (2.22) we can determine u, v uniquely. Hence η and $\xi + (u + iv)\eta$, where u and v are determined uniquely, are two members of s_L which satisfy (1.5).

(iii) $U_\alpha[\eta] = 0 = U_\alpha[\xi]$: then from (2.4) and (2.5) $a = b = h = c = c' = 0$. (2.23)

From (2.1), using (2.23) we get

$$2u(\alpha_3\xi_1\eta_1 - \alpha_3\xi_2\eta_2 + \alpha_4\xi'_1\eta'_1 - \alpha_4\xi'_2\eta'_2)$$
$$- 2v(\alpha_3\xi_2\eta_1 + \alpha_3\xi_1\eta_2 + \alpha_4\xi'_2\eta'_1 + \alpha_4\xi'_1\eta'_2) = 0, \tag{2.24}$$

and from (2.2), using (2.23) we get

$$2u(\alpha_3\xi_2\eta_1 + \alpha_3\xi_1\eta_2 + \alpha_4\xi'_2\eta'_1 + \alpha_4\xi'_1\eta'_2)$$
$$- 2v(-\alpha_3\xi_1\eta_1 + \alpha_3\xi_2\eta_2 - \alpha_4\xi'_1\eta'_1 + \alpha_4\xi'_2\eta'_2) = 0, \tag{2.25}$$

a) If the co-efficient of (2.24) is not zero then $u = 0$, $v = 0$. The only solutions satisfying the QBC (1.5) are ξ and η.

b) If the co-efficient determinant of (2.24) and (2.25) is zero

ie.
$$\left.\begin{aligned} &\alpha_3(\xi_1\eta_1 - \xi_2\eta_2) + \alpha_4(\xi'_1\eta'_1 - \xi'_2\eta'_2) = 0, \\ &\alpha_3(\xi_2\eta_1 + \xi_1\eta_2) + \alpha_4(\xi'_2\eta'_1 + \xi'_1\eta'_2) = 0, \end{aligned}\right\} \tag{2.26}$$

then (2.24) and (2.25) is satisfied by all non-trivial u, v.

Hence all members of s_L satisfy the QBC (1.5).

3. EXAMPLES

1) Example of case (i)

Let us consider the DE

$$y''(x) + iy'(x) = 0, \quad x \in [0, \pi]. \tag{3.1}$$

Here $\xi(x) = 1$ and $\eta(x) = -i(1 - e^{-ix})$. (3.2)

Here $\xi_1 = 1,\ \xi_2 = 0,\ \xi'_1 = 0,\ \xi'_2 = 0,$

$$\eta_1 = 0,\ \eta_2 = -2,\ \eta'_1 = -1,\ \eta'_2 = 0.$$

We consider the given QBC as

$$\boldsymbol{U_\alpha[y] = y^2(0) + 2y'^2(0) + y^2(\pi) + y'^2(\pi) = 0.} \tag{3.3}$$

Here $U_\alpha[\eta] \neq 0$.

Equations (2.1) and (2.2) become

$$-u^2 + v^2 + 4v + 2 = 0 \tag{3.4}$$

and

$$u(v + 2) = 0. \tag{3.5}$$

Solving (3.4) and (3.5) we get,

If $u = 0$, then $v = -2 + \sqrt{2}$ and $v = -2 - \sqrt{2}$

If $v = -2$, then $u = i\sqrt{2}$ and $u = -i\sqrt{2}$

The members of s_L satisfying the above QBC are

$$1 + \left(-2 + \sqrt{2}\right)\left(1 - e^{-ix}\right) \text{ and } 1 + \left(-2 - \sqrt{2}\right)\left(1 - e^{-ix}\right)$$

2) a) Example of case (ii)

Let us consider the DE (3.1),

and consider the given QBC as

$$\boldsymbol{U_\alpha[y] = y^2(0) + 2y'^2(0) + y^2(\pi) + 2y'^2(\pi) = 0.} \tag{3.6}$$

Here $U_\alpha[\eta] = 0$, $U_\alpha[\xi] \neq 0$,

Equations (2.21) and (2.22) become $2v = -1$ and $v = 0$

Hence, the members of s_L satisfying the above QBC are η and $\xi + (u + iv)\eta$

i.e. $-i(1 - e^{-ix})$ and $½(1 + e^{-ix})$.

b) Let us consider the DE

$$xy''(x) + 2y'(x) + \frac{1}{4}xy(x) = 0,\ x \in [\pi, 2\pi]. \tag{3.7}$$

Here $\xi(x) = \frac{\left(\pi \sin\frac{x}{2}\right)}{x} - \frac{\left(2\cos\frac{x}{2}\right)}{x} = -\frac{\left(2\pi\cos\frac{x}{2}\right)}{x}$

Hence $\xi_1 = \frac{1}{\pi}$, $\xi_2 = 0$, $\xi'_1 = -\frac{1}{4} - \frac{1}{2}\pi^2$, $\xi'_2 = 0$,

$\eta_1 = 1$, $\eta_2 = 0$, $\eta'_1 = -\frac{1}{2}\pi$, $\eta'_2 = 0$

We consider the given QBC as

$$U_\alpha[y] = y^2(\pi) + y'^2(\pi) + y^2(2\pi) - 8\pi^2 y'^2(2\pi) = 0. \tag{3.8}$$

Here $U_\alpha[\eta] = 0$, $U_\alpha[\xi] \neq 0$

Equations (2.21) and (2.22) become

$$u = \frac{\pi^4 + 2\pi^2 + 2}{-4\pi\left(1 + \pi^2\right)} \text{ and } v = 0.$$

Hence, the members of s_L satisfying the above QBC are η and $\xi + (u + iv)\eta$

i.e. $-\frac{\left(2\pi\cos\frac{x}{2}\right)}{x}$ and

$$\frac{\pi\sin\frac{x}{2}}{x} + \frac{\pi^4 - 2\pi^2 - 2}{2\left(\pi^2 + 1\right)}\cos\frac{x}{2}$$

3) Example of case (iii)(a)

Let us consider the DE (3.1),

and consider the given QBC as

$$U_\alpha[y] = -y^2(0) + 2y'^2(0) + y^2(\pi) + 2y'^2(\pi) = 0. \tag{3.9}$$

Here $U_\alpha[\eta] = 0$, $U_\alpha[\xi] = 0$,

From (2.24) and (2.25) we get the co-efficient determinant **not** equal to zero.

Hence $v = 0$ and $u = 0$.

$\therefore$ The members of s_L satisfying the above QBC are ξ and η i.e., 1 and $-i(1 - e^{-ix})$.

4) Example of case (iii) (b)

Let us consider the DE

$$y''(x) + 4y(x) = 0, \quad x \in [0, \pi]. \tag{3.10}$$

Here $\xi(x) = \cos 2x$ and $\eta(x) = \dfrac{1}{2}\sin 2x.$

Hence $\xi_1 = 1,\ \xi_2 = 0, \qquad \xi'_1 = 0,\ \xi'_2 = 0,$

$\eta_1 = 0,\ \eta_2 = 0, \qquad \eta'_1 = 1,\ \eta'_2 = 0,$

and consider the given QBC as

$$U_\alpha[y] = -y^2(0) + 2y'^2(0) + y^2(\pi) - 2y'^2(\pi) = 0 \tag{3.11}$$

Here $U_\alpha[\eta] = 0$, $U_\alpha[\xi] = 0$.

From (2.24) and (2.25) we get the co-efficient determinant equal to zero. Hence (2.24) and (2.25) are satisfied by all non- trivial u, v.

Hence all members of s_L satisfy the above QBC i.e., ξ, η and $\xi + (u + iv)\eta$

i.e., 1, $-i(1 - e^{-ix})$ and $1 + (u + iv)\,[-i(1 - e^{-ix})]$ for all u, v satisfy the given QBC.

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